



A decomposition of Brouwer’s fan theorem

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Abstract: We introduce axioms L_{FAN} and C_{FAN} , where the former follows from the law of excluded middle and the latter follows from the axiom of countable choice. Then we show that Brouwer’s fan theorem is constructively equivalent to $L_{\text{FAN}} + C_{\text{FAN}}$. This decomposition of the fan theorem into a logical axiom and a function existence axiom contributes to the programme of constructive reverse mathematics.

2000 *Mathematics Subject Classification* 03F03, 03F55 (primary); 03F60 (secondary)

Keywords: fan theorem, weak König lemma, constructive mathematics, reverse mathematics

The objective of *constructive reverse mathematics* is to classify theorems by means of logical axioms (that means, statements which follow from the law of excluded middle) and function existence axioms (that means, statements which follow from the axiom of choice). Given a theorem T , one aims at determining a logical axiom L_T and a function existence axiom C_T such that, in the framework of a suitable base system, T is equivalent to $L_T + C_T$. A decomposition of the weak König lemma can be found in Ishihara [3]. In this paper, we provide a classification of Brouwer’s fan theorem.

A suitable framework for carrying out constructive reverse mathematics is a function-based, intuitionistic formal system, like HA^ω . In this system, if $A(n)$ is equivalent to a quantifier-free formula, one may prove:

$$(1) \quad \exists \alpha \in \{0, 1\}^{\mathbb{N}} \forall n (\alpha(n) = 0 \leftrightarrow A(n)),$$

see Troelstra and van Dalen [6, Chapter 9, Proposition 1.8].

Let $\{0, 1\}^*$ denote the set of all finite binary sequences u, v, w . We write $|u|$ for the length of u and $u * v$ for the concatenation of u and v . That means, for $u = (u(0), \dots, u(l-1))$ and $v = (v(0), \dots, v(k-1))$, we have $|u| = l$ and

$$u * v = (u(0), \dots, u(l-1), v(0), \dots, v(k-1)).$$

If $i \in \{0, 1\}$, we write $u * i$ for $u * (i)$ and $i * u$ for $(i) * u$. We use Greek letters for infinite binary sequences. We write $\bar{\alpha}n$ for $(\alpha(0), \dots, \alpha(n-1))$. A subset A of $\{0, 1\}^*$ is

- *detachable* if there exists a function $F : \{0, 1\}^* \rightarrow \mathbb{N}$ such that

$$\forall u (u \in A \leftrightarrow F(u) = 0);$$

(For example, (1) implies that, for every quantifier-free formula $C(n)$, the set $\{n \mid C(n)\}$ is detachable.)

- a Π_1^0 -set if there exists a function $G : \{0, 1\}^* \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall u (u \in A \leftrightarrow \forall k (G(u, k) = 0));$$

- *closed under restriction* if

$$\forall u, v (u * v \in A \rightarrow u \in A);$$

- an *infinite tree* if
 - it is detachable
 - it is closed under restriction
 - $\forall n \exists u (|u| = n \wedge u \in A)$;
- a *spread* if
 - $() \in A$
 - $\forall u (u \in A \rightarrow \exists i \in \{0, 1\} (u * i \in A))$.

A function α is an *infinite path* of A if $\forall n (\bar{\alpha}n \in A)$. The *weak König lemma* reads as follows.

WKL Every infinite tree has an infinite path.

The following axiom is called the *lesser limited principle of omniscience*.

LLPO If a function α has the property

$$\forall n, m (n \neq m \rightarrow (\alpha(n) = 0 \vee \alpha(m) = 0)),$$

then

$$\forall n (\alpha(2n) = 0) \vee \forall n (\alpha(2n + 1) = 0).$$

Note that this axiom is a consequence of the law of excluded middle. The following characterisation of LLPO in terms of trees has also been mentioned in Berger et al. [2].

Lemma 1 *The following axioms are equivalent.*

- (1) LLPO

(2) If T is an infinite tree, then either

$$T_0 = \{u \mid 0 * u \in T\}$$

or

$$T_1 = \{u \mid 1 * u \in T\}$$

is an infinite tree.

Proof “1. \Rightarrow 2.” Assume LLPO and let T be an infinite tree. Define α by

$$\alpha(2n) = 1 \stackrel{\text{def}}{\Leftrightarrow} \forall u (|u| = n \rightarrow 0 * u \notin T)$$

and

$$\alpha(2n + 1) = 1 \stackrel{\text{def}}{\Leftrightarrow} \forall u (|u| = n \rightarrow 1 * u \notin T).$$

Furthermore, define β by

$$\beta(n) = 1 \stackrel{\text{def}}{\Leftrightarrow} \alpha(n) = 1 \wedge \forall k < n (\alpha(k) = 0).$$

By LLPO, there is an $i \in \{0, 1\}$ such that $\forall n (\beta(2n + i) = 0)$. We show that T_i is an infinite tree. Suppose that there is an m such that $i * w \notin T$ for all w with $|w| = m$. Then $\alpha(2m + i) = 1$. Since T is an infinite tree, we have $\forall k (\alpha(2k + (1 - i)) = 0)$. Therefore, there exists an $n \leq m$ such that $\beta(2n + i) = 1$. This contradiction shows that T_i is indeed an infinite tree.

“2. \Rightarrow 1.” Assume that the function γ has the property

$$\forall n, m (n \neq m \rightarrow (\gamma(n) = 0 \vee \gamma(m) = 0)).$$

Define an infinite tree T by

$$u \in T \stackrel{\text{def}}{\Leftrightarrow} \exists i \in \{0, 1\} \forall k < |u| (u(k) = i \wedge \gamma(2k + i) = 0).$$

If T_i is an infinite tree, then

$$\forall n (\gamma(2n + i) = 0).$$

□

The following statement is a version of the axiom of countable choice.

C_{WKL} Every Π_1^0 -spread has an infinite path.

The following decomposition of WKL can be found in Ishihara [3]. We recall it, because we want to compare it with the main result of this paper, the decomposition of FAN. We even give a proof of it, because there is a slight difference between C_{WKL} and the choice axiom used in Ishihara [3].

Proposition 2

$$\text{WKL} \leftrightarrow \text{LLPO} + \text{C}_{\text{WKL}}$$

Proof Assume WKL. First, we show LLPO by applying Lemma 1. Let T be an infinite tree. By WKL, there exists an infinite path α of T . Set $i = \alpha(0)$. Then

$$\{w \mid i * w = \bar{\alpha}(|w| + 1)\} \subseteq T_i.$$

Therefore, T_i is an infinite tree.

Next, we show C_{WKL} . To this end, let A be Π_1^0 -spread. There is a function $G : \{0, 1\}^* \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall u (u \in A \leftrightarrow \forall k (G(u, k) = 0)).$$

Define an infinite tree T by

$$u \in T \stackrel{\text{def}}{\Leftrightarrow} \forall k, l \leq |u| (G(\bar{u}k, l) = 0).$$

Then there exists an infinite path α of T , which is also an infinite path of A .

Now assume both LLPO and C_{WKL} . Let T be an infinite tree. By LLPO and Lemma 1 we obtain that the set A , given by

$$u \in A \stackrel{\text{def}}{\Leftrightarrow} \forall n \exists w (|w| = n \wedge u * w \in T),$$

is a spread. Therefore, by C_{WKL} , there is an infinite path α of A . This function α is also an infinite path of T . \square

A detachable subset B of $\{0, 1\}^*$ is

- *closed under extension* if $\forall u, v (u \in B \rightarrow u * v \in B)$;
- *a bar* if $\forall \alpha \exists n (\bar{\alpha}n \in B)$;
- *a uniform bar* if $\exists N \forall \alpha \exists n \leq N (\bar{\alpha}n \in B)$.

Brouwer's fan theorem for detachable bars reads as follows.

FAN Every bar is a uniform bar.

The following Lemma can be found in Ishihara [4].

Lemma 3 *The following statements are equivalent.*

- FAN
- *For every bar B which is closed under extension there exists an N such that $\forall u (|u| = N \rightarrow u \in B)$.*

- Every bar which is closed under extension is a uniform bar.

Berger and Ishihara [1] have shown that the statement ‘every infinite tree with at most one infinite path has an infinite path’ is constructively equivalent to FAN. See also Schwichtenberg [5] for a more formal proof of this result. This characterisation of FAN, together with Proposition 2, gives rise to the definition of the axioms L_{FAN} and C_{FAN} .

A subset A of $\{0, 1\}^*$ has *at most one infinite path* if

$$\forall \alpha, \beta (\exists n (\alpha(n) \neq \beta(n)) \rightarrow \exists m (\bar{\alpha}m \notin A \vee \bar{\beta}m \notin A)).$$

L_{FAN} If T is an infinite tree with at most one infinite path, then either

$$T_0 = \{u \mid 0 * u \in T\}$$

or

$$T_1 = \{u \mid 1 * u \in T\}$$

is an infinite tree.

C_{FAN} If A is a Π_1^0 -spread and B is a bar, then there exists a $u \in A \cap B$.

The axioms introduced so far are related as follows.

Lemma 4 $LLPO$ implies L_{FAN} and C_{WKL} implies C_{FAN} .

Proof The fact that $LLPO$ implies L_{FAN} follows from Lemma 1. In order to prove the second implication, assume C_{WKL} and fix a bar B . Assume further that A is a Π_1^0 -spread. By C_{WKL} , there is an infinite path α of A . Since B is a bar, there exists an n such that $\bar{\alpha}n \in B$. Thus $\bar{\alpha}n \in A \cap B$. \square

Now we are ready to prove the decomposition of FAN.

Proposition 5

$$\text{FAN} \leftrightarrow L_{\text{FAN}} + C_{\text{FAN}}$$

Proof First we show that FAN implies L_{FAN} . Let T be an infinite tree with at most one infinite path. This implies

$$\forall \alpha, \beta \exists n (0 * \bar{\alpha}n \notin T \vee 1 * \bar{\beta}n \notin T).$$

Fix α and define a bar B by

$$B = \{u \mid 0 * \bar{\alpha}|u| \notin T \vee 1 * u \notin T\}.$$

Since B is closed under extension, Lemma 3 implies the existence of an n such that $\forall u (|u| = n \rightarrow u \in B)$. Thus we have

$$\forall \alpha \exists n \forall \beta (0 * \bar{\alpha}n \notin T \vee 1 * \bar{\beta}n \notin T).$$

Define another bar B' by

$$B' = \{v \mid \forall \beta (0 * v \notin T \vee 1 * \bar{\beta}|v| \notin T)\}.$$

Again, since B' is closed under extension, Lemma 3 implies the existence of an n such that $\forall v (|v| = n \rightarrow v \in B')$. We obtain

$$(2) \quad \exists n \forall \alpha, \beta (0 * \bar{\alpha}n \notin T \vee 1 * \bar{\beta}n \notin T).$$

Since T is an infinite tree, it contains an element u of length $n + 1$. If $u(0) = i$, then (2) implies that

$$\forall v (|v| = n \rightarrow (1 - i) * v \notin T).$$

But T is an infinite tree, therefore, T_i must be an infinite tree as well.

Next, we show that FAN implies C_{FAN} . Let A be a Π_1^0 -spread. Then for every n there is a u with $|u| = n$ such that every restriction of u belongs to A . Let B be a bar. By FAN , there is an N such that

$$\forall u (|u| = N \rightarrow \exists n \leq N (\bar{u}n \in B)).$$

Suppose that u is of length N and that every restriction of u belongs to A . Then there is an $n \leq N$ such that $\bar{u}n \in A \cap B$.

Finally, we show that the combination of L_{FAN} and C_{FAN} implies FAN . Let B be a bar which is closed under extension. Define

$$v \prec u \stackrel{def}{\iff} |v| < |u| \vee (|v| = |u| \wedge \exists k < |v| (\bar{v}k = \bar{u}k \wedge v(k) < u(k))).$$

If $() \in B$, then B is a uniform bar. Assume now that $() \notin B$ and define a subset P of $\{0, 1\}^*$ by

$$u \in P \stackrel{def}{\iff} u \notin B \wedge \forall v (u \prec v \rightarrow v \in B).$$

A sequence u belongs to P if and only if it is the largest element of $\{0, 1\}^* \setminus B$, with respect to the ordering \prec . Since B is closed under extension, we can conclude that $u \in P$ if and only if

$$u \notin B \wedge \forall v (|v| \leq |u| + 1 \wedge u \prec v \rightarrow v \in B).$$

Therefore, P is detachable. Since \prec is a total relation on $\{0, 1\}^*$, we can conclude that P has at most one element, that means

$$\forall u, v (u \neq v \rightarrow u \notin P \vee v \notin P).$$

Furthermore, if there exists an element u of P , then every w with $|w| = |u| + 1$ belongs to B , which implies that B is a uniform bar. Define an infinite tree T by

$$u \in T \stackrel{\text{def}}{\Leftrightarrow} u \notin B \vee \exists n < |u| (\bar{u}n \in P \wedge \forall l (n \leq l < |u| \rightarrow u(l) = 0)).$$

Note that here we use the assumption that $() \notin B$, because $B = \{0, 1\}^*$ would imply that $T = \emptyset$. If a sequence u belongs to T , then either it does not belong to B or else some proper restriction of u belongs to P . We show that T has at most one infinite path. Fix α and β and suppose that there is an m such that $\alpha(m) \neq \beta(m)$. Since B is both a bar and closed under extension, there is an n such that $\bar{\alpha}n \neq \bar{\beta}n$, $\bar{\alpha}n \in B$, and $\bar{\beta}n \in B$. We will derive a contradiction from the assumption that both $\bar{\alpha}n$ and $\bar{\beta}n$ are in T . By the definition of T , there are $k, l < n$ such that

- $\bar{\alpha}k \in P$;
- $\alpha(k) = \alpha(k + 1) = \dots = \alpha(n - 1) = 0$;
- $\bar{\beta}l \in P$;
- $\beta(l) = \beta(l + 1) = \dots = \beta(n - 1) = 0$.

If $\bar{\alpha}k = \bar{\beta}l$, then $\bar{\alpha}n = \bar{\beta}n$. If $\bar{\alpha}k \neq \bar{\beta}l$, then $\bar{\alpha}k$ and $\bar{\beta}l$ are two distinct elements of P . Either case leads to a contradiction. Therefore, T has at most one infinite path. Define a subset T' of T by

$$u \in T' \stackrel{\text{def}}{\Leftrightarrow} \{w \mid u * w \in T\} \text{ is an infinite tree.}$$

Since we have

$$\forall u (u \in T' \leftrightarrow \forall m \exists w (|w| = m \wedge u * w \in T)),$$

we can conclude that T' is a Π_1^0 -set. Next, we show that T' is a spread. Since T is an infinite tree, $() \in T'$. Assume that $u \in T'$. Then

$$T_u = \{w \mid u * w \in T\}$$

is an infinite tree. By L_{FAN} there exists $i \in \{0, 1\}$ such that $(T_u)_i$ is an infinite tree, which implies that $u * i$ is in T' . This concludes the proof that T' is a spread. Now C_{FAN} yields the existence of a u such that $u \in T' \cap B$. Since we have $T' \subseteq T$, we even obtain that $u \in B \cap T$. By the definition of T , some restriction of u lies in P . Therefore, B is a uniform bar. \square

Overall, we obtain the following picture.

$$\begin{array}{ccc} \text{WKL} & \leftrightarrow & \text{LLPO} + \text{C}_{\text{WKL}} \\ & & \downarrow \qquad \downarrow \\ \text{FAN} & \leftrightarrow & \text{L}_{\text{FAN}} + \text{C}_{\text{FAN}} \end{array}$$

Acknowledgment While writing this paper, the author was being supported by a ‘Feodor Lynen Return Fellowship’, provided by the Alexander von Humboldt-Foundation.

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Received: 30 January 2009 Revised: 22 April 2009