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Clarke's Generalized Gradient and Edalat's L-derivative

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Abstract: Clarke [2, 3, 4] introduced a generalized gradient for real-valued Lipschitz continuous functions on Banach spaces. Using domain theoretic notions, Edalat [5, 6] introduced a so-called L-derivative for real-valued functions and showed that for Lipschitz continuous functions Clarke's generalized gradient is always contained in this L-derivative and that these two notions coincide if the underlying Banach space is finite dimensional. He asked whether they coincide as well if the Banach space is infinite dimensional. We show that this is the case.

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1 Introduction

Clarke [4, Page 1] observed that "nonsmooth phenomena in mathematics and optimization occur naturally and frequently, and there is a need to be able to deal with them. We are thus led to study differential properties of nondifferentiable functions." Motivated by this observation, Clarke [2, 3, 4] introduced a generalized gradient $\partial f(x)$ for real-valued Lipschitz continuous functions f on Banach spaces. Using domain theoretic notions in the realm of computable analysis, Edalat [5, 6] introduced a so-called L-derivative for real-valued functions on Banach spaces. Domain theory arose in the context of computer science and logic. While the primary application of domain theory is in the semantics of programming languages, domain theoretic notions can also be applied successfully in computable analysis, as shown in [5, 6] and in other articles by Edalat. As Clarke's generalized gradient and Edalat's L-derivative are defined using rather different mathematical notions it is remarkable that they are closely connected. Edalat [5, 6] showed that for Lipschitz continuous functions Clarke's generalized gradient is always contained in his L-derivative and that these two notions coincide if the underlying Banach space is finite dimensional. He asked whether they coincide as well if the Banach space is infinite dimensional. We show that, indeed, for Lipschitz

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continuous real functions on arbitrary Banach spaces they coincide. In view of the fact that both notions are defined within different mathematical theories, the fact that they coincide for Lipschitz continuous functions may be taken as further indication that the resulting notion is a natural one. That they coincide also shows that the domain theoretic definition by Edalat can be used for a computability theoretic treatment of Clarke's generalized gradient.

In the following section we introduce some basic notions and collect fundamental facts about a Banach space X, its dual space X^* and the weak* topology on X^*. Then, in Section 3, for a Lipschitz continuous function on X, we introduce Clarke's generalized directional derivative and Clarke's generalized gradient. In the following section, for an open subset U of X, we introduce a function that may be considered as a "global" version of Clarke's generalized directional derivative. Then we prove a crucial property of this function. Using this function, in Section 5 we introduce a certain weak* compact subset of X^* that contains Clarke's generalized gradient. In Section 6 we formulate several fundamental facts concerning continuous functions $f : X \to Y$ where X may be an arbitrary topological space, eg a Banach space, and where Y is a directed complete partial order (dcpo). Furthermore, we consider some special bounded complete dcpo's. In Section 7 we introduce Edalat's so-called ties of functions that are needed for the definition of his L-derivative. We also show how they are related to the weak* compact subsets introduced in Section 5. Finally, in Section 8 we introduce Edalat's L-derivative and show that for a Lipschitz continuous function on an arbitrary Banach space it coincides with Clarke's generalized gradient.

2 Basic notions

In this section, we remind the reader of several well-known notions concerning a Banach space X, its dual space X^* and the weak* topology on X^*. We will consider only vector spaces over $\mathbb{R}$, the field of real numbers.

Let X be a real Banach space with norm $||\cdot||$. For any $x \in X$ and $r > 0$ let

$$\mathrm{B}(x, r) := \{y \in X \ : \ ||y - x|| < r\}$$

be the open ball with radius r and midpoint x. For subsets $S, T \subseteq X$ we write as usual

$$S + T := \{x \in X \ : \ (\exists s \in S)\, (\exists t \in T)\, x = s + t\}.$$

Let X^* be the linear vector space of all continuous linear functions $\zeta : X \to \mathbb{R}$. With the norm $||\cdot||_*$ defined by

$$||\zeta||_* := \sup\{|\zeta(v)| \ : \ v \in X, ||v|| \leq 1\}$$

this space is a Banach space as well. As either $|\zeta(v)| = \zeta(v)$ or $|\zeta(v)| = -\zeta(v) = \zeta(-v)$ and $||-v|| = ||v||$ we may write as well

$$||\zeta||_* = \sup\{\zeta(v) \ : \ v \in X, ||v|| \leq 1\}.$$

The weak* topology is a topology on the set X^*. It is defined to be the coarsest topology such that for any $x \in X$ the function

$$l_x\colon \ X^* \to \mathbb{R} \text{ defined by } l_x(\zeta) := \zeta(x)$$

is continuous. Every subset $U \subseteq X^*$ that is open in the weak* topology is also open in the topology induced by the norm $|| \cdot ||_*$. The converse is in general not true; see, eg, Megginson [9, Theorem 2.6.2 and Corollary 2.6.3]. It is well known that X^* with the weak* topology is a topological vector space and a Hausdorff space; see, eg, Rudin [10, Page 66 and Theorem 1.12]. Subsets of X^* that are compact in the weak* topology will be called *weak* compact*.

3 Clarke's Generalized Gradient

The terminology in this section is copied from Clarke [4, Chapter 2].

Let X be a Banach space with norm $|| \cdot ||$. Let Y be a subset of X, and let c be a non-negative real number. A function $f\colon \ Y \to \mathbb{R}$ is called *Lipschitz continuous with Lipschitz constant c* if for all $x, y \in Y$

$$|f(x) - f(y)| \leq c \cdot ||x - y||.$$

A function $f\colon \ X \to \mathbb{R}$ is called *Lipschitz continuous with Lipschitz constant c near a point $x \in X$* if there is an $\varepsilon > 0$ such that the restriction of f to the ball $\mathrm{B}(x, \varepsilon)$ is Lipschitz continuous with Lipschitz constant c. A function $f\colon \ X \to \mathbb{R}$ is called *Lipschitz continuous (near x)* if there exists a real number $c \geq 0$ such that f is Lipschitz continuous with Lipschitz constant c (near x).

Let $f\colon \ X \to \mathbb{R}$ be Lipschitz continuous near some point $x \in X$. Fix an arbitrary $v \in X$. The *generalized directional derivative of f at x in the direction v*, denoted $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) := \limsup_{z \to x, t \downarrow 0} \frac{f(z + tv) - f(z)}{t}.$$

In the following section we spell this out in more detail (Lemma 1(4)).

For any $x \in X$ the *generalized gradient of f at x*, denoted $\partial f(x)$, is defined as follows:

$$\partial f(x) = \{\zeta \in X^* \ : \ (\forall v \in X)\ \zeta(v) \leq f^\circ(x; v)\}.$$

This is a nonempty, convex and weak* compact subset of X^*; see Clarke [4, Proposition 2.1.2(a)].

4 A Global Version of Clarke's Generalized Directional Derivative

Let X be a Banach space with norm $||\cdot||$. For a nonempty, open set $U \subseteq X$ and a function $f : \mathrm{dom}(f) \subseteq X \to \mathbb{R}$ with $U \subseteq \mathrm{dom}(f)$ that is Lipschitz continuous on U we define

$$\tilde{f}(U,v) := \sup\left\{\frac{f(z+tv)-f(z)}{t} \ :\ z \in U,\ t > 0, z+tv \in U\right\}$$

for $v \in X$. We show in the following lemma that this is well defined. This function can be considered as a global version of Clarke's generalized directional derivative. In the following lemma several elementary assertions about this function are collected.

Lemma 1 *Let X be a Banach space, let $U \subseteq X$ be a nonempty open subset, and let $f : \mathrm{dom}(f) \subseteq X \to \mathbb{R}$ with $U \subseteq \mathrm{dom}(f)$ be a function Lipschitz continuous on U. For the first four of the following five assertions, fix an arbitrary $v \in X$.*

(1) *The value $\tilde{f}(U,v)$ is well defined, and if $c \geq 0$ is a Lipschitz constant for f on U then $\tilde{f}(U,v) \leq c \cdot ||v||$.*

(2) *If $U' \subseteq U$ is a nonempty open subset of U then $\tilde{f}(U',v) \leq \tilde{f}(U,v)$.*

(3) *If $x \in U$ then $f^\circ(x,v) \leq \tilde{f}(U,v)$.*

(4) *If $x \in U$ then $\lim_{n\to\infty} \tilde{f}(B(x,2^{-n}),v) = f^\circ(x,v)$.*

(5) *The function $v \mapsto \tilde{f}(U,v)$ is positively homogeneous, i.e, for all $r > 0$ and $v \in X$,*

$$\tilde{f}(U,rv) = r\tilde{f}(U,v).$$

Proof Fix some $v \in X$.

(1) Let $c \geq 0$ be a Lipschitz constant for f on U. Then for all $z \in U$ and $t > 0$ such that $z+tv \in U$ we have

$$\frac{f(z+tv)-f(z)}{t} \leq \frac{|f(z+tv)-f(z)|}{t} \leq \frac{ct||v||}{t} = c \cdot ||v||.$$

This shows that $\tilde{f}(U,v)$ is well defined and satisfies $\tilde{f}(U,v) \leq c \cdot ||v||$.

(2) This follows directly from the definition of $\tilde{f}(U,v)$.

(3) This is a consequence of the definitions of $f^\circ(x,v)$ and of $\tilde{f}(U,v)$.

(4) Note that by the second statement the sequence $(\tilde{f}(B(x,2^{-n}),v))_{n\in\mathbb{N}}$ is non-increasing, and by the third statement it is bounded from below. Therefore its limit exists. That its limit is equal to $f^\circ(x,v)$ is simply a restatement of the definition of $f^\circ(x,v)$.

(5) We additionally fix some real number $r>0$. Then

$$\begin{aligned}\tilde{f}(U,rv) &= \sup\left\{\frac{f(z+trv)-f(z)}{t} \;:\; z\in U,\ t>0, z+trv\in U\right\}\\ &= \sup\left\{r\cdot\frac{f(z+sv)-f(z)}{s} \;:\; z\in U,\ s>0, z+sv\in U\right\}\\ &= r\cdot\sup\left\{\frac{f(z+sv)-f(z)}{s} \;:\; z\in U,\ t>0, z+sv\in U\right\}\\ &= r\cdot\tilde{f}(U,v).\end{aligned}$$ □

Clarke's generalized directional derivative is positively homogeneous and subadditive [4, Proposition 2.1.1]. In Lemma 1(5) we have seen that also the global function $v\mapsto\tilde{f}(U,v)$ is positively homogeneous. In the following lemma we show that it is subadditive as well if U is convex.

Lemma 2 *Let X be a Banach space, let $U\subseteq X$ be a convex open subset, and $f:\mathrm{dom}(f)\subseteq X\to\mathbb{R}$ with $U\subseteq\mathrm{dom}(f)$ be a function Lipschitz continuous on U. Then the function $v\mapsto\tilde{f}(U,v)$ is subadditive, ie, for all $v,w\in X$,*

$$\tilde{f}(U,v+w)\le\tilde{f}(U,v)+\tilde{f}(U,w).$$

Proof We will fix some $\varepsilon>0$ and show that for any $v,w\in X$

$$\tilde{f}(U,v)+\tilde{f}(U,w)\ge\tilde{f}(U,v+w)-\varepsilon.$$

Once this is proved for any $\varepsilon>0$, the assertion

$$\tilde{f}(U,v)+\tilde{f}(U,w)\ge\tilde{f}(U,v+w)$$

follows.

So, let us fix some $\varepsilon>0$ and elements $v,w\in X$. We choose a point $z\in U$ and a real number $t>0$ such that $z+t(v+w)\in U$ and

$$\frac{f(z+t(v+w))-f(z)}{t}\ge\tilde{f}(U,v+w)-\varepsilon.$$

One might try to prove the assertion simply by replacing the quotient on the left hand side by the following term:

$$\frac{f(z+t(v+w))-f(z+tv)}{t}+\frac{f(z+tv)-f(z)}{t}.$$

If $z+tv$ would happen to be an element of U, then this term would be a lower bound for $\tilde{f}(U,w)+\tilde{f}(U,v)$, and we would be done. Unfortunately, the point $z+tv$ may lie outside of the open set U; see Figure 1. We will have to proceed differently. This is

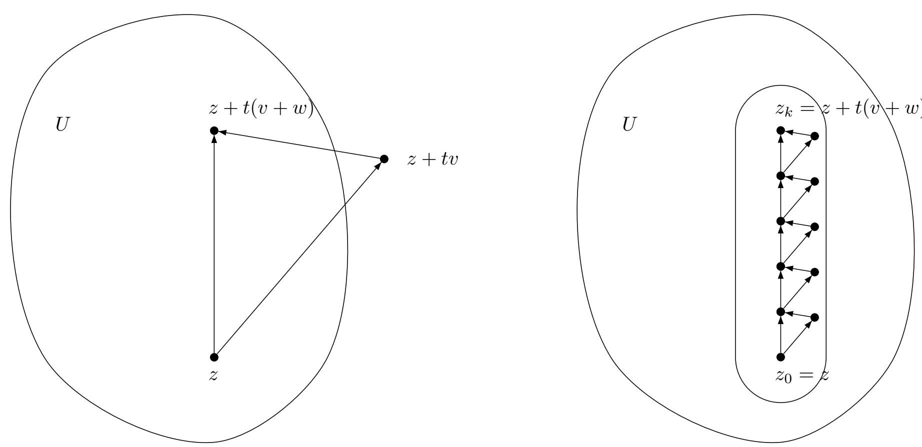

Figure 1: The diagram on the left hand side illustrates a proof attempt for Lemma 2 that does not work. The diagram on the right hand side illustrates the construction in the proof for $k=5$.

where the convexity of U will be important. As U is convex, the line segment

$$L:=\{z+r(v+w)\ :\ 0\leq r\leq t\}$$

is a subset of U. Since this line segment L is compact and U is open there exists some $\delta>0$ such that even the set

$$L+B(0,\delta)$$

is a subset of U. Let k be a positive integer so large that

$$\frac{t||v||}{k}<\delta.$$

Then for $i=0,\ldots,k$ the points

$$z_i:=z+i\cdot\frac{t}{k}\cdot(v+w)$$

are elements of the line segment L and, thus, of the open set U, and the points $z_i+\frac{t}{k}v$ are elements of $L+B(0,\delta)$, and, hence, elements of the open set U as well; see Figure 1.

Using these points we can replace the quotient $\frac{f(z+t(v+w))-f(z)}{t}$ by a sum of quotients as follows:

$$\begin{aligned}
&\frac{f(z+t(v+w))-f(z)}{t} \\
=\ & \frac{f(z_k)-f(z_0)}{t} \\
=\ & \sum_{i=0}^{k-1} \frac{f(z_{i+1})-f(z_i)}{t} \\
=\ & \sum_{i=0}^{k-1} \frac{f(z_i+\frac{t}{k}(v+w))-f(z_i)}{t} \\
=\ & \sum_{i=0}^{k-1} \left(\frac{f(z_i+\frac{t}{k}(v+w))-f(z_i+\frac{t}{k}v)}{t} + \frac{f(z_i+\frac{t}{k}v)-f(z_i)}{t} \right) \\
=\ & \sum_{i=0}^{k-1} \frac{1}{k} \cdot \frac{f(z_i+\frac{t}{k}(v+w))-f(z_i+\frac{t}{k}v)}{t/k} + \sum_{i=0}^{k-1} \frac{1}{k} \cdot \frac{f(z_i+\frac{t}{k}v)-f(z_i)}{t/k} \\
\leq\ & \sum_{i=0}^{k-1} \frac{1}{k} \cdot \tilde{f}(U,w) + \sum_{i=0}^{k-1} \frac{1}{k} \cdot \tilde{f}(U,v) \\
=\ & \tilde{f}(U,w) + \tilde{f}(U,v).
\end{aligned}$$

We have shown

$$\tilde{f}(U,v) + \tilde{f}(U,w) \geq \tilde{f}(U,v+w) - \varepsilon,$$

where $\varepsilon > 0$ was chosen arbitrarily. This ends the proof. □

5 About Nonempty, Convex, Weak* Compact Sets

Let X be a Banach space. In this section, first we observe that any nonempty, convex, weak* compact subset K of X^* can be expressed with the help of its support function. Then, using the global version of the generalized directional derivative, we introduce certain nonempty, convex, weak* star compact subsets of X^*.

Let K be a nonempty, weak* compact subset of X^*. It is well known (see, eg, Megginson [9, Corollary 2.6.9]) that any such set is bounded, ie, there is some non-negative real number B such that

$$(\forall \zeta \in K)\ ||\zeta||_* \leq B.$$

This implies for all $\zeta \in K$ and $v \in X$

$$\zeta(v) \leq ||\zeta||_* \cdot ||v|| \leq B \cdot ||v||.$$

Thus, for $v \in X$, by

$$\mathrm{s}_K(v) := \sup\{\zeta(v) \ : \ \zeta \in K\}$$

a real number is defined that satisfies

$$\mathrm{s}_K(v) \leq B \cdot ||v||.$$

The function $\mathrm{s}_K : \ X \to \mathbb{R}$ is called the *support function* of K.

Lemma 3 *Let X be a Banach space. Let $K \subseteq X^*$ be a nonempty, convex, and weak* compact subset of X^*. Then*

$$K = \{\zeta \in X^* \ : \ (\forall v \in X)\ \zeta(v) \leq \mathrm{s}_K(v)\}.$$

Proof The inclusion "$\subseteq$" follows from the definition of $\mathrm{s}_K(v)$. We wish to prove the inclusion "$\supseteq$". Therefore, let us consider some $\xi \in X^*$ with $\xi \notin K$. By the second part of Beer [1, Theorem 1.4.2] there exist an element $v \in X$ and a real number α such that either

$$\xi(v) > \alpha \text{ and } (\forall \zeta \in K)\ \zeta(v) < \alpha \tag{1}$$

or

$$\xi(v) < \alpha \text{ and } (\forall \zeta \in K)\ \zeta(v) > \alpha.$$

In the second case we obtain

$$\xi(-v) = -\xi(v) > -\alpha \text{ and } (\forall \zeta \in K)\ \zeta(-v) = -\zeta(v) < -\alpha,$$

thus, (1) holds for $-v$ and $-\alpha$. Therefore, we can assume that (1) holds. Then

$$\mathrm{s}_K(v) \leq \alpha < \xi(v)$$

and, hence,

$$\xi \notin \{\zeta \in X^* \ : \ (\forall v \in X)\ \zeta(v) \leq \mathrm{s}_K(v)\}.$$ □

Let $U \subseteq X$ be a nonempty, open subset of the Banach space X, and let $f : \mathrm{dom}(f) \subseteq X \to \mathbb{R}$ with $U \subseteq \mathrm{dom}(f)$ be a function Lipschitz continuous on U. We define

$$K_{U,f} := \{\zeta : X \to \mathbb{R} \ : \ \zeta \text{ is linear and } (\forall v \in X)\ \zeta(v) \leq \tilde{f}(U, v)\}.$$

Lemma 4 (1) *$K_{U,f}$ is a nonempty, convex, and weak* compact subset of X^*.*

(2) *For all $x \in U$, $\partial f(x) \subseteq K_{U,f}$.*

(3) *Let n_0 be a nonnegative integer with $B(x, 2^{-n_0}) \subseteq U$. Then*

$$\partial f(x) = \bigcap_{n \geq n_0} K_{B(x,2^{-n}),f}.$$

Proof (1) Let $c \geq 0$ be a Lipschitz constant for f on U. According to Lemma 1(1) for any $v \in X$ we have $\tilde{f}(U, v) \leq c \cdot ||v||$. Hence, for $\zeta \in K_{U,f}$, $\zeta(v) \leq c \cdot ||v||$ and $-\zeta(v) = \zeta(-v) \leq c \cdot || - v|| = c \cdot ||v||$, thus $|\zeta(v)| \leq c \cdot ||v||$. This implies $\zeta \in X^*$, and $K_{U,f}$ is contained in the set $\{\zeta \in X^* : ||\zeta||_* \leq c\}$. By Alaoglu's theorem (see, eg, Megginson [9, Theorem 2.6.18]) this set is weak* compact. As $K_{U,f}$ is a weak* closed subset of this set, $K_{U,f}$ is itself weak* compact; see, eg, Engelking [7, Theorem 3.1.2]. It is also clear that $K_{U,f}$ is a convex set. That $K_{U,f}$ is nonempty follows from the second assertion and the fact that $\partial f(x)$ is nonempty.

(2) This follows from Lemma 1(3).

(3) By the second assertion we obtain $\partial f(x) \subseteq K_{B(x,2^{-n}),f}$ for any $n \geq n_0$, hence

$$\partial f(x) \subseteq \bigcap_{n \geq n_0} K_{B(x,2^{-n}),f}.$$

For the inverse inclusion consider some $\zeta \in \bigcap_{n \geq n_0} K_{B(x,2^{-n}),f}$. Then, for all $n \geq n_0$ and all $v \in X$,

$$\zeta(v) \leq \tilde{f}(B(x, 2^{-n}), v),$$

hence,

$$\zeta(v) \leq \lim_{n \to \infty} \tilde{f}(B(x, 2^{-n}), v) = f^\circ(x, v)$$

(compare Lemma 1(4)), and this implies $\zeta \in \partial f(x)$. That was to be shown. □

6 Continuous Functions from an Arbitrary Topological Space to a Bounded Complete DCPO

It is the purpose of this section to provide several fundamental facts concerning continuous functions $f\colon X \to Y$ where X may be an arbitrary topological space, eg a Banach space, and where Y is a directed complete partial order (dcpo), in particular, where Y is a bounded complete dcpo. Furthermore, we consider some special bounded complete dcpo's. We will need all this later in order to define Edalat's L-derivative for arbitrary functions. In order to make the presentation self-contained we introduce basic notions about dcpo's as well.

A set Z with a binary relation $\sqsubseteq \subseteq Z \times Z$ satisfying the following three conditions

(1) $(\forall z)\, z \sqsubseteq z$ (*reflexivity*),

(2) $(\forall x, y, z)\, (x \sqsubseteq y \wedge y \sqsubseteq z) \Rightarrow x \sqsubseteq z$ (*transitivity*),

(3) $(\forall y, z)\, (y \sqsubseteq z \wedge z \sqsubseteq y) \Rightarrow y = z$ (*antisymmetry*),

is called a *partial order*. Let $(Z, \sqsubseteq)$ be a partial order.

- An element $z \in Z$ is called an *upper bound* of a subset $S \subseteq Z$ if $(\forall s \in S)\, s \sqsubseteq z$.
- An element $z \in Z$ is called a *supremum* or *least upper bound* of a subset $S \subseteq Z$ if it is an upper bound of S and if for all upper bounds y of S one has $z \sqsubseteq y$. Obviously, if a set S has a supremum, then this supremum is unique. Then we denote it by $\sup(S)$.
- A subset $S \subseteq Z$ is called *directed* if it is nonempty and for any two elements $x, y \in S$ there exists an upper bound $z \in S$ of the set $\{x, y\}$.
- Z is called a *dcpo* if for any directed subset $S \subseteq Z$ there exists a supremum of S in Z.
- A subset $S \subseteq Z$ is called *upwards closed* if for any elements $s, z \in Z$: if $s \in S$ and $s \sqsubseteq z$ then $z \in S$.

The following lemma is well known.

Lemma 5 (See, eg, Goubault-Larrecq [8, Proposition 4.2.18]) *Let $(Z, \sqsubseteq)$ be a partial order. The set of all subsets $O \subseteq Z$ satisfying the following two conditions:*

(1) *O is upwards closed,*

(2) *if $S \subseteq Z$ is a directed subset with $\sup(S) \in O$ then $S \cap O \neq \emptyset$,*

is a topology on Z, called the Scott topology.

Consider now some partial order $(Z, \sqsubseteq)$ and an arbitrary topological space X. We call a total function $f\colon X \to Z$ *Scott continuous* if it is continuous with respect to the given topology on X and the Scott topology on Z. Let $\mathrm{C}(X, Z)$ denote the set of all Scott continuous functions $f\colon X \to Z$. On this set we define a binary relation $\sqsubseteq_{\mathrm{C}}$ by

$$f \sqsubseteq_{\mathrm{C}} g :\Longleftrightarrow (\forall x \in X)\, f(x) \sqsubseteq g(x).$$

Proposition 6 *Let $(Z, \sqsubseteq)$ be a dcpo, and let X be an arbitrary topological space. Then $\mathrm{C}(X, Z)$ with $\sqsubseteq_{\mathrm{C}}$ is a dcpo. Furthermore, if $F \subseteq \mathrm{C}(X, Z)$ is a $\sqsubseteq_{\mathrm{C}}$-directed set then the function $g\colon X \to Z$ defined by*

$$g(x) := \sup(\{f(x) \ : \ f \in F\})$$

is Scott continuous and the least upper bound of F.

Proof It is clear that $(\mathrm{C}(X,Z), \sqsubseteq_{\mathrm{C}})$ is a partial order.

Now we show that $\mathrm{C}(X,Z)$ with $\sqsubseteq_{\mathrm{C}}$ is even a dcpo. Let $F \subseteq \mathrm{C}(X,Z)$ be a $\sqsubseteq_{\mathrm{C}}$-directed set. Then for every $x \in X$, the set

$$F(x) := \{f(x) \;:\; f \in F\} = \{z \in Z \;:\; (\exists f \in F)\, z = f(x)\}$$

is $\sqsubseteq$-directed. We define a total function $g\colon\; X \to Z$ by $g(x) := \sup(F(x))$.

First, we claim that this function g is Scott continuous. Fix some point $x \in X$, and let $O \subseteq Z$ be a Scott open set with $g(x) \in O$. We have to show that there is an open set $U \subseteq X$ with $x \in U$ and $g(U) \subseteq O$. From $\sup(F(x)) = g(x) \in O$ and from the assumption that $F(x)$ is a directed set we conclude $F(x) \cap O \neq \emptyset$, hence, there is some $f \in F$ with $f(x) \in O$. As f is Scott continuous, there is some open set $U \subseteq X$ with $x \in U$ and $f(U) \subseteq O$. Thus, for all $y \in U$, on the one hand we have $f(y) \in O$, and on the other hand, by definition of g, $f(y) \sqsubseteq g(y)$. As O is Scott open and, thus, upwards closed, we obtain $g(y) \in O$. This shows $g(U) \subseteq O$. Thus, we have shown that g is Scott continuous.

By the definition of g, for all $f \in F$ we have $f \sqsubseteq_{\mathrm{C}} g$, thus g is an upper bound of F. Finally, if h is an arbitrary upper bound of F, then for all $x \in X$, $g(x) = \sup(F(x)) \leq h(x)$. Hence, g is a least upper bound of F. □

Let $(Z, \sqsubseteq)$ be a partial order.

- An element $y \in Z$ is called a *least element* of Z if for all $z \in Z$, $y \sqsubseteq z$. Obviously, if a least element exists then it is unique. Usually, when a least element exists, it is denoted $\perp$.
- A subset $S \subseteq Z$ is called *bounded* if there exists an upper bound $z \in Z$ for S.
- Z is called *bounded complete* if for any bounded subset $S \subseteq Z$ there exists a supremum of S in Z.

Lemma 7 *Any bounded complete partial order has a least element.*

This is well known. For completeness sake we give the proof.

Proof In any partial order, every element is an upper bound of the empty set. Hence, in a bounded complete partial order $\sup(\emptyset)$ exists. It is a least element. □

The following proposition covers the cases of bounded complete dcpos that we will need.

Proposition 8 *Let Y be a nonempty topological vector space whose topology is a Hausdorff topology. Let*

$$Z := \{Y\} \cup \{K \ : \ K \subseteq Y \text{ is a nonempty, convex, compact set}\}.$$

(1) *Then Z with $\sqsubseteq$ defined as reverse inclusion is a bounded complete dcpo with least element Y.*

(2) *If $S \subseteq Z$ is bounded or directed then*

$$\sup(S) = \bigcap_{K \in S} K.$$

Proof First, we observe that $(Z, \sqsubseteq)$ is a partial order.

Next, we prove the following claim.

Claim 1: For any $S \subseteq Z$, the set $\bigcap_{K \in S} K$ is either empty or an element of Z.

For the proof, we distinguish three cases.

Case I: $S = \emptyset$. Then $\bigcap_{K \in S} K = Y$, and this is an element of Z.

Case II: $S = \{Y\}$. Then again $\bigcap_{K \in S} K = Y$, and this is an element of Z.

Case III: S contains at least one element K_0 that is different from Y, ie, that is nonempty, convex and compact. As all elements of S are closed (any compact subset of a Hausdorff space is closed; see, eg, Engelking [7, Theorem 3.1.8]) and the intersection of arbitrarily many closed sets is closed as well, the intersection $\bigcap_{K \in S} K$ is a closed subset of Y. In fact, it is a closed subset of the compact set K_0, and hence (see, eg, [7, Theorem 3.1.2]) compact itself. Note also that all elements of S are convex and that the intersection of arbitrarily many convex sets is again a convex set. Thus, $\bigcap_{K \in S} K$ is a convex and compact subset of Y. If this intersection is not empty then it is an element of Z. We have proved Claim 1.

We continue with the following claim.

Claim 2: If $S \subseteq Z$ is bounded or directed then $\bigcap_{K \in S} K$ is not empty.

Let $S \subseteq Z$ be a bounded set, and let $K_0 \in Z$ be an upper bound of S. This means $K_0 \subseteq K$ for all $K \in S$, hence, $K_0 \subseteq \bigcap_{K \in S} K$. And as all elements of Z are nonempty, K_0 is nonempty. Hence, $\bigcap_{K \in S} K$ is nonempty. Now let $S \subseteq Z$ be directed. If $S = \{Y\}$ then $\bigcap_{K \in S} K = Y$, and this set is nonempty. Let us assume that S contains at least one element K_0 that is different from Y, hence, K_0 is a nonempty, convex, compact subset of Y. Using the assumption that S is a directed set, one shows by induction that the set

$$S' := \{K \cap K_0 \ : \ K \in S\}$$

has the finite intersection property, ie, the intersection of any finite subset of S' is nonempty. As the elements of S' are closed subsets of the compact set K_0, this implies that $\bigcap_{K'\in S'} K' \neq \emptyset$; see, eg, [7, Theorem 3.1.1]. It is clear that $\bigcap_{K\in S} K = \bigcap_{K'\in S'} K'$. We have proved Claim 2.

Now let $S \subseteq Z$ be bounded or directed. Claims 1 and 2 imply that $\bigcap_{K\in S} K$ is an element of Z. It is clear that $\bigcap_{K\in S} K$ is an upper bound of S. On the other hand, any upper bound of S must be a subset of any K in S, hence, a subset of $\bigcap_{K\in S} K$. Thus, this intersection is the least upper bound of S. ◻

Example 9 Let us apply Proposition 8 to $Y = \mathbb{R}$. Hence, let

$$\mathcal{I} := \{\mathbb{R}\} \cup \{[a,b] \ : \ a,b \in \mathbb{R}, a \leq b\}$$

be the set whose elements are all nonempty, compact intervals and the whole set of real numbers. We define $\sqsubseteq$ as reverse inclusion:

$$A \sqsubseteq B :\iff B \subseteq A.$$

Then $(\mathcal{I}, \sqsubseteq)$ is a bounded complete dcpo with least element $\mathbb{R}$. The supremum of a bounded or directed set $S \subseteq \mathcal{I}$ is the intersection $\bigcap_{I\in S} I$.

Example 10 Let X be a Banach space. We apply Proposition 8 to $Y := X^*$ with the weak* topology. It is well known that X^* with the weak* topology is a topological vector space and a Hausdorff space; see, eg, Rudin [10, Page 66]. We define Z_{convex} by

$$Z_{\text{convex}} := \{X^*\} \cup \{K \ : \ K \subseteq X^* \text{ is a nonempty, convex, weak}^*\text{compact set}\}.$$

On Z_{convex} we define $\sqsubseteq$ again as reverse inclusion. Then $(Z_{\text{convex}}, \sqsubseteq)$ is a bounded complete dcpo with least element X^*. The supremum of a bounded or directed set $S \subseteq \mathcal{I}$ is the intersection $\bigcap_{K\in S} K$.

Edalat [5, Section 3], [6, Section 4] defined his L-derivate as the supremum of a certain class of "elementary step functions" from a Banach space X to Z_{convex} from Example 10. These elementary step functions are Scott continuous. We will see that under suitable conditions the supremum of a certain class of such functions exists in the dcpo of Scott continuous functions.

Definition 11 Let X be an arbitrary topological space and $(Z, \sqsubseteq)$ be a partial order with least element $\perp$. For any open subset $U \subseteq X$ and any $z \in Z$ we define the (total) function $(U \searrow z) : X \to Z$ by

$$(U \searrow z)(x) := \begin{cases} z & \text{if } x \in U, \\ \perp & \text{if } x \notin U. \end{cases}$$

We call such a function an *elementary step function*.

It is easy to see that any such function is Scott continuous. In fact, we will need the following stronger statement, which is an immediate consequence of Goubault-Larrecq [8, Lemma 5.7.10].

Lemma 12 *Let X be an arbitrary topological space and $(Z, \sqsubseteq)$ be a partial order with least element $\bot$. Let F be a finite set of elementary step functions from X to Z such that for each $x \in X$ the set $\{f(x) \ : \ f \in F\}$ has a least upper bound. Then the function $g\colon\ X \to Z$ defined by*

$$g(x) := \sup(\{f(x) \ : \ f \in F\})$$

is Scott continuous.

Note that this lemma implies that every elementary step function is Scott continuous.

Proof This is an immediate consequence (almost a reformulation) of [8, Lemma 5.7.10]. □

For the formulation of the desired result the following notion is useful. We call a set F of total functions from X to Z *pointwise bounded* if for every $x \in X$, the set

$$F(x) := \{f(x) \ : \ f \in F\} = \{z \in Z \ : \ (\exists f \in F)\, z = f(x)\}$$

is $\sqsubseteq$-bounded.

Theorem 13 *Let X be an arbitrary topological space, and let $(Z, \sqsubseteq_Z)$ be a bounded complete dcpo. Let F be a set of elementary step functions from X to Z. If F is pointwise bounded then the total function $g\colon\ X \to Z$ defined by*

$$g(x) := \sup\{f(x) \ : \ f \in F\}$$

is Scott continuous and a least upper bound of F.

Proof Let F be a pointwise bounded set of elementary step functions. Remember that we have already seen that every elementary step function is continuous, ie, $F \subseteq C(X, Z)$. Let the total function $g\colon\ X \to Z$ be defined by

$$g(x) := \sup\{f(x) \ : \ f \in F\}.$$

This function is well defined because we assume that F is pointwise bounded and Z is bounded complete. We have to show that g is Scott continuous and a least upper bound of F. In fact, once we have shown that g is Scott continuous it is clear that g is a least upper bound of F. Thus, we are now going to show that g is Scott continuous.

For any finite subset $E \subseteq F$, the set $\{f(x) \; : \; f \in E\}$ is bounded because F is pointwise bounded. As Z is bounded complete, $\sup\{f(x) \; : \; f \in E\}$ exists. According to Lemma 12, the function $g_E : X \to Z$ defined by

$$g_E(x) := \sup\{f(x) \; : \; f \in E\}$$

is Scott continuous. It is clear that it is a least upper bound of E, ie, $g_E = \sup(E)$. It is straightforward to see that the set

$$D := \{g_E \; : \; E \text{ is a finite subset of } F\}$$

is a directed subset of $C(X, Z)$. By Proposition 6 $\sup(D)$ exists and is a Scott continuous function. Finally, it is also straightforward to see that $\sup(D) \sqsubseteq_C g$ and $g \sqsubseteq_C \sup(D)$, thus, $\sup(D) = g$. As $\sup(D)$ is Scott continuous, the proof is finished. □

7 Ties of Functions

Let X be a Banach space. As in Example 10 we define Z_{convex} by

$$Z_{\text{convex}} := \{X^*\} \cup \{K \; : \; K \subseteq X^* \text{ is a nonempty, convex, weak}^* \text{compact set}\}.$$

On Z_{convex} we define $\sqsubseteq$ again as reverse inclusion. In Example 10 we already mentioned that $(Z_{\text{convex}}, \sqsubseteq)$ is a bounded complete dcpo with least element X^*. Note that for any $K \in Z_{\text{convex}}$ and any $x \in X$ the set

$$K(x) := \{r \in \mathbb{R} \; : \; (\exists \zeta \in K) \; r = \zeta(x)\}$$

is an element of $\mathcal{I}$ as defined in Example 9. Let $V \subseteq X$ be a nonempty open subset of X. Following Edalat [5, Definition 1], [6, Definition 3.1], for a nonempty open set $U \subseteq V$ and an element $K \in Z_{\text{convex}}$, we call the set

$$\begin{aligned}\delta_V(U, K) := \{f \; : \; & f\colon \; \text{dom}(f) \to \mathbb{R} \text{ is a function with } U \subseteq \text{dom}(f) \subseteq V \\ & \text{and } (\forall x, y \in U) \; K(x - y) \sqsubseteq f(x) - f(y)\}\end{aligned}$$

the *single tie of O with K*. Here, the formula $K(x-y) \sqsubseteq f(x)-f(y)$ has to be understood with respect to the dcpo of Example 9. We can rewrite this definition without using domain theoretic language as follows:

$$\begin{aligned}\delta_V(U, K) = \{f \; : \; & f\colon \; \text{dom}(f) \to \mathbb{R} \text{ is a function with } U \subseteq \text{dom}(f) \subseteq V \\ & \text{and } (\forall x, y \in U)(\exists \zeta \in K) \, f(x) - f(y) = \zeta(x - y)\}.\end{aligned}$$

Remark 14 Actually, Edalat [5, Definition 1], [6, Definition 3.1] considered only convex open sets U. Convexity of U will be important later. But since the definition of $\delta(U, K)$ makes sense for arbitrary open U and since we wish to show where convexity of U will be important, right now we do not restrict ourselves to convex U.

Lemma 15 *Let $V \subseteq X$ be a nonempty open subset of X, and fix some $K \in Z_{\text{convex}}$.*

(1) *For any nonempty open sets $U_1, U_2 \subseteq V$*

$$U_1 \subseteq U_2 \Longrightarrow \delta_V(U_1, K) \supseteq \delta_V(U_2, K).$$

(2) *Let U be a nonempty open subset of V, and let $K \in Z_{\text{convex}}$ be different from X^*, ie, let K be a nonempty, convex, weak* compact subset of X^*. If $f \in \delta_V(U, K)$ then*

(a) *f is Lipschitz continuous on U and*

(b) *$K_{U,f} \subseteq K$.*

Proof (1) This is clear.

(2) (a) We already mentioned that any nonempty, convex, weak* compact subset of X^* is bounded, that is, there exists some $c \geq 0$ such that for all $\zeta \in K$, $||\zeta||_* \leq c$. Consider arbitrary $x, y \in U$. Due to the assumption $f \in \delta_V(U, K)$ there exists some $\zeta \in K$ with $\zeta(x-y) = f(x) - f(y)$. We obtain

$$|f(x) - f(y)| = |\zeta(x-y)| \leq ||\zeta||_* \cdot ||x-y|| \leq c \cdot ||x-y||.$$

This shows that f is Lipschitz continuous on U.

(b) Due to the assumption $f \in \delta_V(U, K)$, for any $z \in U$ and $t > 0$ with $z + tv \in U$ there exists some $\zeta \in K$ with

$$\frac{f(z+tv) - f(z)}{t} = \frac{\zeta(z+tv-z)}{t} = \zeta(v).$$

This implies

$$\tilde{f}(U, v) \leq \mathrm{s}_K(v).$$

By Lemma 3 we obtain

$$K_{U,f} \subseteq K. \qquad \square$$

Note that in the following lemma we consider convex U.

Lemma 16 *Let $V \subseteq X$ be a nonempty open subset of X. Let U be a nonempty convex open subset of V. Let $f\colon \ \mathrm{dom}(f) \to \mathbb{R}$ with $U \subseteq \mathrm{dom}(f) \subseteq V$ be a function that is Lipschitz continuous on U. Then $f \in \delta_V(U, K_{U,f})$.*

Proof We have to show that for every $x, y \in U$ there exists some $\zeta \in K_{U,f}$ with $f(x) - f(y) = \zeta(x - y)$. Consider some $x, y \in U$.

If $x = y$ then we choose an arbitrary $\zeta \in K_{U,f}$ (remember that by Lemma 4(1) $K_{U,f}$ is not empty) and obtain

$$f(x) - f(y) = 0 = \zeta(0) = \zeta(x - y).$$

Now we consider the case $x \neq y$. We set $w := x - y$. Let W be the one-dimensional subspace of X generated by w, and let the linear function $\zeta_0 : W \to \mathbb{R}$ be defined by $\zeta_0(\alpha \cdot w) := \alpha \cdot (f(x) - f(y))$, for any $\alpha \in \mathbb{R}$. Then for all $\alpha \geq 0$

$$\begin{aligned}
\zeta_0(\alpha w) &= \alpha \cdot (f(x) - f(y)) \\
&= \alpha \cdot \frac{f(y + 1 \cdot w) - f(y)}{1} \\
&\leq \alpha \cdot \tilde{f}(U, w) \\
&= \tilde{f}(U, \alpha w),
\end{aligned}$$

and for $\alpha < 0$

$$\begin{aligned}
\zeta_0(\alpha w) &= \alpha \cdot (f(x) - f(y)) \\
&= (-\alpha) \cdot (f(y) - f(x)) \\
&= (-\alpha) \cdot \frac{f(x + 1 \cdot (-w)) - f(x)}{1} \\
&\leq (-\alpha) \cdot \tilde{f}(U, -w) \\
&= \tilde{f}(U, (-\alpha) \cdot (-w)) \\
&= \tilde{f}(U, \alpha w).
\end{aligned}$$

Hence, for all $v \in W$ we have $\zeta_0(v) \leq \tilde{f}(U, v)$. Now remember that the function $v \mapsto \tilde{f}(U, v)$, mapping X to $\mathbb{R}$, is positively homogeneous (Lemma 1(5)) and subadditive (Lemma 2; this is where the convexity of U is used). By the Hahn Banach Extension Theorem (see, eg, Megginson [9, Theorem 1.9.5]) there exists a linear function $\zeta : X \to \mathbb{R}$ satisfying

(1) $\zeta(v) = \zeta_0(v)$ for all $v \in W$ and

(2) $\zeta(v) \leq \tilde{f}(U, v)$ for all $v \in X$.

The first equation implies

$$\zeta(x - y) = \zeta(w) = \zeta_0(w) = f(x) - f(y).$$

The second equation implies that ζ is an element of $K_{U,f}$. □

8 Edalat's L-derivative

In this section we consider the setting of Example 10, that is, X is a Banach space, and on

$$Z_{\text{convex}} := \{X^*\} \cup \{K \ : \ K \subseteq X^* \text{ is a nonempty, convex, weak}^*\text{compact set}\}$$

we define $\sqsubseteq$ as reverse inclusion. Then, $(Z_{\text{convex}}, \sqsubseteq)$ is a bounded complete dcpo with least element X^*; see Example 10.

Note that in the following lemma we do not assume that the open set U considered there is convex.

Lemma 17 *Let X be a Banach space, $V \subseteq X$ a nonempty open subset of X, $U \subseteq V$ a nonempty open subset of V, $K \in Z_{\text{convex}}$, and $f : V \to \mathbb{R}$ a function with $f \in \delta_V(U,K)$. If f is Lipschitz continuous near some point $x \in V$ then*

$$\partial f(x) \subseteq (U \searrow K)(x).$$

Proof If $x \notin U$ then $(U \searrow K)(x) = X^*$, and $\partial f(x) \subseteq X^*$ is clear. Let us consider the case $x \in U$. Then $(U \searrow K)(x) = K$. Thus, we have to show $\partial f(x) \subseteq K$. This is clear if $K = X^*$. Let us assume that K is not equal to X^*, hence, that K is a nonempty, convex, weak* compact subset of X^*. According to Lemma 15(2), f is Lipschitz continuous on U and $K_{U,f} \subseteq K$. According to Lemma 4(2) we have $\partial f(x) \subseteq K_{U,f}$. Together we obtain $\partial f(x) \subseteq K$. □

Let X be a Banach space, $V \subseteq X$ a nonempty open subset of X, and $f : V \to \mathbb{R}$ an arbitrary function. We define:

$$D(f) := \{(U,K) \ : \ U \subseteq V \text{ is nonempty, open, and convex and } K \in Z_{\text{convex}} \text{ and } f \in \delta_V(U,K)\}.$$

First, we note that $D(f)$ is not empty. For example, if $U \subseteq V$ is an open ball (balls are convex) then (U, X^*) is an element of $D(f)$ (this is shown by an application of the Hahn Banach Extension Theorem in a similar manner as in the proof of Lemma 16).

Lemma 18 *The set*

$$F := \{(U \searrow K) \ : \ (U,K) \in D(f)\}$$

is pointwise bounded.

Proof It is sufficient to show that for each $x \in X$ the set $F(x)$ is a bounded subset of Z_{convex}. We distinguish two cases.

Case I: There is no pair $(U, K) \in D(f)$ with $x \in U$ and $K \neq X^*$. Then $F(x) = \{X^*\}$. This set is bounded by X^* itself, the least element of Z_{convex}.

Case II: There is some pair (U_0, K_0) with $x \in U_0$ and $K_0 \neq X^*$. Then K_0 is a nonempty, convex, weak* compact subset of X^*. According to Lemma 15(2), f is Lipschitz continuous on U_0 and, hence, Lipschitz continuous near x. By Lemma 17 $\partial f(x) \subseteq (U \searrow K)(x)$ for any $(U, K) \in D(f)$. This shows that $F(x)$ is bounded by $\partial f(x)$. □

Now we can define Edalat's L-derivative [5, Section 3], [6, Section 4].

Definition 19 Let X be a Banach space, $V \subseteq X$ be a nonempty open subset of X, and $f : V \to \mathbb{R}$ be an arbitrary function. The function $\mathcal{L}f$ defined by

$$\mathcal{L}(f) := \sup\{(U \searrow K) \ : \ (U, K) \in D(f)\}$$

is called *L-derivative* of f.

According to Lemma 18 and Theorem 13 this function $\mathcal{L}f$ is well defined, Scott continuous and a least upper bound of the set $\{(U \searrow K) \ : \ (U, K) \in D(f)\}$. Note that we do not make any assumption about the function f. The following result describes the L-derivative via Clarke's generalized gradient.

Theorem 20 *Let X be a Banach space, and let $V \subseteq X$ be a nonempty open subset of X. Let $f : V \to \mathbb{R}$ be an arbitrary function. Fix some point $x \in V$.*

(1) *If f is not Lipschitz continuous near x then $\mathcal{L}f(x) = X^*$.*

(2) *If f is Lipschitz continuous near x then $\mathcal{L}f(x) = \partial f(x)$.*

Since we have already done most of the work, the proof is fairly short.

Proof Let $D(f)$ be defined as before Lemma 18. We have already seen that $D(f)$ is not empty. According to Theorem 13, Lemma 18 and the last assertion in Example 10

$$\mathcal{L}f(x) = \sup\{(U \searrow K)(x) \ : \ (U, K) \in D(f)\} = \bigcap_{(U,K) \in D(f)} (U \searrow K)(x).$$

(1) Let us assume that f is not Lipschitz continuous near x. Then there cannot exist a pair $(U,K) \in D(f)$ with nonempty, convex, compact $K \subseteq X^*$ because otherwise, according to Lemma 15(2), f were Lipschitz continuous on U and, hence, Lipschitz continuous near x. We conclude that $(U \searrow K)(x) = X^*$ for all $(U,K) \in D(f)$. Hence, also
$$\mathcal{L}(f)(x) = X^*.$$

(2) Let us assume that f is Lipschitz continuous near x. On the one hand, according to Lemma 17,
$$\partial f(x) \subseteq \bigcap_{(U,K)\in D(f)} (U \searrow K)(x) = \mathcal{L}f(x).$$
For the other direction, note that there is some n_0 such that $B(x,2^{-n_0}) \subseteq V$ and f is Lipschitz continuous on $B(x,2^{-n_0})$. Then, according to Lemma 4(3),
$$\partial f(x) = \bigcap_{n\geq n_0} K_{B(x,2^{-n}),\,f}.$$
And by Lemma 16 (remember that balls in X are convex), for all $n \geq n_0$ we have $f \in \delta_V(B(x,2^{-n}), K_{B(x,2^{-n}),f})$. This implies $(B(x,2^{-n}), K_{B(x,2^{-n}),f}) \in D(f)$, hence,
$$\mathcal{L}f(x) \subseteq \bigcap_{n\geq n_0} K_{B(x,2^{-n}),\,f}.$$
We have shown $\mathcal{L}f(x) \subseteq \partial f(x)$ as well. □

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Note added in proof

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