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Genericity and UD–random reals

WESLEY CALVERT
JOHANNA N.Y. FRANKLIN

Abstract: Avigad introduced the notion of UD–randomness based in Weyl's 1916 definition of uniform distribution modulo one. We prove that there exists a weakly 1–random real that is neither UD–random nor weakly 1–generic. We also show that no 2–generic real can Turing compute a UD–random real.

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1 Introduction

In [1] Avigad introduced a randomness notion based on uniform distribution modulo 1, as described by Weyl in [10]. A sequence $\langle x_i \rangle$ of real numbers is said to be *uniformly distributed modulo one* in the case that for every interval $I \subseteq [0,1]$, we have

$$\lim_{n\to\infty} \frac{|\{i < n \mid [x_i] \in I\}|}{n} = \mu(I)$$

where μ is Lebesgue measure and $[x_i]$ denotes the fractional part of x_i. Weyl proved that if $\langle a_i \rangle$ is a sequence of distinct integers, then for almost every real x, the sequence $\langle a_i x \rangle$ is uniformly distributed modulo one [10].

The definition below considers recursive sequences $\langle a_i \rangle$ of distinct integers, observing that there are countably many such sequences and that therefore almost every $x \in [0,1]$ has the property that $\langle a_i \rangle$ is uniformly distributed modulo one:

Definition 1.1 [1] A real $x \in [0,1]$ is UD–random if and only if for every recursive sequence $\langle a_i \rangle$ of distinct integers, $\langle a_i x \rangle$ is uniformly distributed modulo one.

We know several facts about the relationship between UD–randomness and other randomness notions, all due to Avigad's original paper. For instance, every Schnorr random real is UD–random; however, there are UD–random reals that are not weakly 1–random [1]. This is surprising, since weak 1–randomness is such a weak randomness

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notion: not only is every Schnorr random weakly 1–random, but so (by a result of Kurtz) is every weakly 1–generic [6]. Avigad observed that, in consequence, there are weakly 1–random reals that are not UD–random, since a UD–random must satisfy the law of large numbers, and no real that is even weakly 1–generic does.

In this paper we investigate the relationship of the UD–random reals to the weakly 1–random reals and the weakly 1–generic reals further, and determine when a generic real can Turing compute a UD–random. There are very few conclusions that can be drawn about Turing degrees of UD–randoms from existing results. We can conclude, for instance, by a result of Nies, Stephan, and Terwijn, that every high Turing degree contains a UD–random because every high Turing degree contains a Schnorr random real [8] and that there is therefore a low UD–random and a hyperimmune-free UD–random using various basis theorems. However, we cannot use any of our standard results involving the coincidence of randomness notions on given classes of Turing degrees. For instance, in the hyperimmune-free degrees, weak 1–randomness is equivalent to Martin-Löf-randomness [8], but we cannot conclude that this also holds of UD–randomness, since there are UD–randoms that are not weakly 1–random. We also know that Schnorr and Martin-Löf randomness coincide outside the high degrees [8], but we cannot conclude that UD–randomness coincides with these notions there as well.

We describe the weakly 1–random degrees further using UD–randomness in Section 2, and in Section 3 we discuss the relationship between the Turing degrees of UD–randoms and generic reals. Section 4 is dedicated to open questions.

1.1 Basic observations and notation

Our notation is standard and follows Soare [9] and Downey and Hirschfeldt [3]; we add that we denote the concatenation of two finite binary strings σ and τ by $\sigma^\frown\tau$.

Since we often discuss finite binary strings in the context of uniform distribution modulo one, we make the following convention: We identify a binary string σ with the binary representation of a real on the unit interval. Of course, a finite string also denotes a real number — a dyadic rational number, in fact (the finite string σ represents the same real as $\sigma^\frown\overline{0}$). From time to time we will apply certain functions (especially multiplication) which will take σ outside the unit interval, but since these functions are always postcomposed with the fractional part function, no ambiguity will result.

Fact 1.2 A real X is UD–random if and only if it is UD–random with respect to dyadic intervals.

Proof One implication is obvious. For the other, suppose X is not UD–random. Suppose, in particular, that for an interval I and for arbitrarily large n we have

$$\left|\frac{|\{i < n \mid [x_i] \in I\}|}{n} - \mu(I)\right| > 2\epsilon.$$

Now we can find I' such that I' has dyadic endpoints and $\mu(I \triangle I') < \epsilon$. Then for arbitrarily large n we have

$$\left|\frac{|\{i < n \mid [x_i] \in I'\}|}{n} - \mu(I')\right| > \epsilon.$$

□

2 Categorizing weakly 1–random reals

Since the UD–random reals and the weakly 1–generic reals are disjoint, one can reasonably ask whether the UD–random weakly 1–random reals and the weakly 1–generic reals partition the weakly 1–random reals or whether there is a weakly 1–random real that is neither UD–random nor weakly 1–generic. We answer this question with the following theorem.

Theorem 2.1 *There is a weakly 1–random real that is neither UD–random nor weakly 1–generic.*

Proof To construct such an $X \in 2^\omega$, we must build not only X itself but also a dense Σ^0_1 set of strings S that X avoids (witnessing that X is not weakly 1–generic) and an infinite recursive sequence $\langle a_i \rangle$ witnessing that X is not UD–random with respect to the interval $(\frac{1}{2}, 1)$. We must also satisfy the following requirement for each Σ^0_1 set W_e to ensure that X is weakly 1–random:

$$\mathcal{R}_e\text{: If } \mu([W_e]) = 1, \text{ then } X \in W_e.$$

Without loss of generality, we take W_e to be a subset of $2^{<\omega}$ rather than ω.

This finite injury construction requires us to keep track of two sequences of finite binary strings for each e: $\langle \sigma_{e,s} \rangle_s$ and $\langle \tau_{e,s} \rangle_s$. Each $\tau_{e,s}$ is a string that is our guess for X under the assumption that no requirement that has higher priority than $\mathcal{R}_e$ will ever act after stage s, and the $\sigma_{e,s}$ are extensions of the $\tau_{e,s}$ that we construct in tandem with the a_i-sequence to ensure that X will not be UD–random. We also build an auxiliary sequence $\langle n_{e,s} \rangle_s$ that will keep track of the measure we need to activate requirement $\mathcal{R}_e$

at stage s, and we will also record the length of the a_i-sequence at the end of each stage and denote this by ℓ_s.

In general, our strategy is this. At stage s, we have a list of the requirements with indices below s that have already acted. For every other index $e < s$, we will keep track of those $e' < e$ such that $\mathcal{R}_e$ has already acted, and we will have an approximation $\sigma_{e,s-1}$ to X based on this list. If our list already includes all $e' < e$ such that $\mathcal{R}_{e'}$ will act at any point, then our guess will be correct. We will work in the cylinder above $\sigma_{e,s-1}$ and wait for a lower-priority W_e to cover a certain fraction of this cylinder. If this happens, we will act to satisfy $\mathcal{R}_e$ and extend our $\sigma_{e,s-1}$ to a string $\tau_{e,s}$ to meet W_e. Then we will extend $\tau_{e,s}$ to $\sigma_{e,s}$ and extend our sequence of a_is accordingly to make sure that $\frac{1}{2} < [a_i\sigma_{e,s}] < 1$ for no more than one-fourth of all a_i in our sequence. To build a dense set of strings S that X avoids, we will add strings to S every time a new requirement acts to ensure that each string at a new, higher level is extended by an element of S. To this end, we will also keep track of the number of times we have acted to satisfy a requirement: we define t_s to be this number at the end of each stage s.

Construction Set $S_0 = \emptyset$ and $\langle a_i \rangle = \langle \rangle$, and let $\sigma_{e,0} = \langle \rangle$ for all e, and $\ell_0 = t_0 = 0$.

At stage s, suppose we have a list $0 \le e_0 < e_1 < e_2 < \ldots < e_k < s$ such that $\mathcal{R}_{e_i}$ has acted already for each i. First, determine whether there is an e less than s such that

$$\mu([W_{e,s}] \cap [\sigma_{e,s-1}]) > \frac{1}{2}\mu([\sigma_{e,s-1}]) \tag{1}$$

and $\mathcal{R}_e$ is not currently satisfied. If there is not, let $S_s = S_{s-1}$ and set $\sigma_{e,s} = \sigma_{e,s-1}$ for all e and go on to the next stage. If there is such an e, choose the least one and act to satisfy $\mathcal{R}_e$.

We begin by choosing a $\tau_{e,s}$ in W_e that extends $\sigma_{e,s-1}$ and avoids S_{s-1}; we choose the leftmost shortest such string. We check to see whether $\frac{|\{i<\ell_{s-1} \mid \frac{1}{2}<[a_i\tau_{e,s}]<1\}|}{\ell_{s-1}}$ is less than $\frac{1}{4}$. If this is the case, set $\sigma_{e,s} = \tau_{e,s}^\frown 0$ and let $a_{\ell_{s-1}+1} = 2^m$, where m is the position of this last 0. Then the fractional part of $a_{\ell_{s-1}+1}\sigma_{e,s}$ is less than $\frac{1}{2}$, and, since $\ell_s = \ell_{s-1} + 1$, it follows that $\frac{|\{i<\ell_s \mid \frac{1}{2}<[a_i\tau_{e,s}]<1\}|}{\ell_s} < \frac{1}{4}$ as well.

On the other hand, if $\frac{|\{i<\ell_{s-1} \mid \frac{1}{2}<[a_i\tau_{e,s}]<1\}|}{\ell_{s-1}}$ is not less than $\frac{1}{4}$, find the least N such that $\frac{|\{i<\ell_{s-1} \mid \frac{1}{2}<[a_i\tau_{e,s}]<1\}|}{\ell_{s-1}+N} < \frac{1}{4}$. Take $\sigma_{e,s} = \tau_{e,s}^\frown 0^N$ and set $a_{\ell_{s-1}+i} = 2^{\ell_s+i}$ for all $1 \le i \le N$. For each i such that $\ell_{s-1} < i \le \ell_{s-1} + N$, the fractional part of $a_i\sigma_{e,s}$ is less than $\frac{1}{4}$, so $\frac{|\{i<\ell_s \mid \frac{1}{2}<[a_i\tau_{e,s}]<1\}|}{\ell_s} < \frac{1}{4}$ once more.

At this point we say that $\mathcal{R}_e$ has been met. However, we are not done with stage s: we must extend our set S to ensure its density. To do this, consider each string of length $|\sigma_{e,s}|$. If such a string has an extension in S_{s-1}, we need not extend it again; otherwise, extend it to a string τ long enough that the measure of $[\tau]$ is no more than $\frac{1}{2^{t_{s-1}+2}} \cdot \mu([\sigma_{e,s}])$. This guarantees that the total measure that S consumes above each string of length $|\sigma_{e,s}|$ is no more than half the available measure.

At this point, we are ready to end this stage. We have already determined $\sigma_{e,s}$. For $e' > e$, reinitialize $\sigma_{e',s}$ by setting it equal to $\sigma_{e,s}$. If we acted to satisfy a requirement at this stage, define $t_s = t_{s-1} + 1$.

Let $X = \lim_{e,s} \sigma_{e,s}$.

Verification

Lemma 2.2 *The limit $X = \lim_e \lim_s \sigma_{e,s}$ exists.*

Proof We proceed by induction. Consider e, and let s_0 be the first stage such that all $e' < e$ that will ever act already have. If we never act to satisfy $\mathcal{R}_e$, then $\sigma_{e,s_0} = \sigma_{e,t}$ for all $t > s_0$. If we act to satisfy $\mathcal{R}_e$ at a later stage s_1, then we never reinitialize σ_{e,s_1} and we have $\sigma_{e,s_1} = \sigma_{e,t}$ for all $t > s_1$. Thus, $\sigma_e = \lim_s \sigma_{e,s}$ exists. Since in our construction, whenever $e_1 > e$, we have $\sigma_e \prec \sigma_{e_1}$, the limit $X = \lim_e \sigma_e$ exists as well. □

Lemma 2.3 *Each requirement $\mathcal{R}_e$ is satisfied and thus X is weakly 1–random.*

Proof If $\mu([W_e]) \neq 1$, then $\mathcal{R}_e$ is satisfied trivially. Suppose $\mu([W_e]) = 1$, and suppose that s_0 is the first stage such that all $e' < e$ that will ever act already have. After s_0, the string $\sigma_{e,s}$ will never be initialized again, so $\sigma_{e,s_0} = \sigma_{e,t}$ for all $t \geq s_0$. Since $\mu([W_e]) = 1$, there will be some stage $s_1 \geq s_0$ such that $\mu([W_{e,s}] \cap [\sigma_{e,s-1}]) > \frac{1}{2}\mu([\sigma_{e,s-1}])$. At this stage, we act to satisfy $\mathcal{R}_e$ and will define σ_{e,s_1} so $[\sigma_{e,s_1}]$ is contained in $[W_e]$. After this stage, $\sigma_{e,s}$ will never be initialized again, so we will have $\sigma_e = \sigma_{e,s_1}$, and since $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \ldots$, we have $X \in [W_e]$. □

Lemma 2.4 *X is not weakly 1–generic.*

Proof We first show that the set of strings S we constructed is a Σ^0_1 dense set as required. It is clear from the construction that it is Σ^0_1. Furthermore, at each stage s at which we add strings to S, we extend all the strings of length $|\sigma_{e,s}|$ for the appropriate e. Since the lengths of the $\sigma_{e,s}$ that we work with are unbounded, we will extend every element of $2^{<\omega}$ by an element in S at some point.

Now we show that X does not meet the Σ^0_1 set of strings S we constructed. We note that the measure of the strings above any $\sigma_{e,s-1}$ is no more than (in fact, is strictly less than)

$$\left(\frac{1}{2^2}+\frac{1}{2^3}+\dots\right)\cdot\mu([\sigma_{e,s-1}])=\frac{1}{2}\cdot\mu([\sigma_{e,s-1}]).$$

Since at each stage at which we set new values of a $\sigma_{e,s}$, we know that Inequality 1 holds, we can see that there is an string τ extending an element of $W_{e,s}$ such that $[\tau]\cap[S_{s-1}]=\emptyset$, and we choose such a string as our $\tau_{e,s}$. Therefore we have $[\sigma_{e,s}]\cap[S_s]=\emptyset$ as well, and since every σ_e avoids S, it follows that X avoids S as well. □

Lemma 2.5 *X is not UD–random.*

Proof We show that the sequence $\langle a_i\rangle$ we built witnesses that X is not UD–random. First, we note that $\langle a_i\rangle$ is recursive and infinite by construction: at every point at which we act to satisfy a requirement, we add at least one more element to this sequence, and since there are infinitely many W_e with measure 1, we act to satisfy a requirement infinitely often. It is clear that $\langle a_i\rangle$ is recursive. Furthermore, it is clear that all the elements of $\langle a_i\rangle$ are distinct: each time we add a new element, it is larger than all the previous terms.

Lemma 2.2 shows us that $X=\lim_e\sigma_e$. Each time we act to meet a requirement $\mathcal{R}_e$ at a stage s, we also define an initial part of $\langle a_i\rangle$ such that $[a_i\sigma_{e,s}]$ is between $\frac{1}{2}$ and 1 for fewer than $\frac{1}{4}$ of the $i<\ell_s$. When this $\sigma_{e,s}$ is the actual value of σ_e, we have a value ℓ_s such that $[a_i\sigma_e,s]=[a_i\sigma_e]=[a_iX]$ for all $i<\ell_s$. Since the values of ℓ_s associated with true values of σ_e form an unbounded sequence, we know that there are infinitely many ℓ_s such that $\frac{|\{i<\ell_s\mid\frac{1}{2}<[a_iX]<1\}|}{\ell_s}<\frac{1}{4}$ and thus X is not UD–random with respect to the sequence $\langle a_i\rangle$ and the interval $(\frac{1}{2},1)$. □

This completes the proof of the Theorem. □

Since our choice of the interval $(\frac{1}{2},1)$ was arbitrary, we have the following result:

Porism 2.6 *If a real X is not UD–random, a recursive sequence can be found witnessing this for any interval $I\subseteq[0,1]$.*

3 Genericity and UD–random reals

There has been a great deal of work relating, on the one hand, Turing degrees of Martin-Löf random, Schnorr random, and recursively random reals, and, on the other, the Turing degrees of generic reals by, for instance, Demuth and Kučera [2], Kautz [5], and Nies, Stephan, and Terwijn [8].

Consider the following question: Given n, is it possible for a (weakly) n–generic real to Turing compute a UD–random even though no (weakly) n–generic real can be UD–random itself? The answer is certainly yes for weak 1–generics, since Kurtz has shown that every hyperimmune degree contains a weakly 1–generic real [6, 7], and there are Schnorr random reals that are hyperimmune. In fact, even a 1–generic real can be Turing equivalent to a UD–random real, since there is a high 1–generic.

Here, we provide a proof that this bound is tight: No 2–generic can Turing compute a UD–random real. Franklin proved in [4] that non-high 1–generics cannot Turing compute Schnorr random reals, which implies that no 2–generic can Turing compute a Schnorr random real. She also proved that no 1–generic can *tt*-compute a Schnorr random real. However, we cannot adapt the first argument to the UD–random case because it relies on the fact that outside the high degrees, Martin-Löf randomness and Schnorr randomness are equivalent. No such result is known for UD–randomness.

Theorem 3.1 *No 2–generic real can Turing compute a UD–random real.*

Proof Let G be a 2–generic real, and let X be a non-recursive real that is Turing reducible to G via a Turing functional Ψ (if X is recursive, then there is nothing to be done). We wish to show that X is not UD–random.

We begin by considering the following statements:

- Ψ^Z is total.
- There are infinitely many n for which $\Psi^Z(n) = 0$.

Each of these is a $\Pi_2^{0,Z}$ statement, so G must force each of these statements to either be true or false. Since $X = \Psi^G$, it follows that G must force the first to be true. Also, X is not recursive and thus not cofinite, so that G must force the second statement to be true as well. Therefore, we may suppose that p is an initial segment of G forcing the truth of both statements. Our forcing conditions will be the set $\mathcal{P} = \{p_i \in 2^{<\omega} \mid p_i \supseteq p\}$. We consider the set $T = \{r_i \mid r_i = \Psi^{p_i}\}$. Since p is enough to force totality, every element r_i in T will have a proper extension r_j in T, so we can interpret T as an infinite recursively enumerable binary tree with X as one of its paths. If a real Ψ^Z is not UD–random, the following statement $P_{\alpha,\beta}$ must be true for some recursive sequence of distinct integers $\langle a_i \rangle$ and the reals α and β (where $0 \leq \alpha < \beta \leq 1$):

$$P_{\alpha,\beta}: \quad (\exists \epsilon > 0)(\forall N)(\exists n > N)\left[\left|\frac{|\{i < n \mid \alpha < [a_i \Psi^Z] < \beta\}|}{n} - (\beta - \alpha)\right| \geq \epsilon\right].$$

To show that X is not UD–random, it will be enough to set $\alpha = \frac{1}{2}$, $\beta = 1$, and $\epsilon = \frac{1}{4}$ and build a recursive sequence $\langle a_i \rangle$ of distinct integers such that for every r_i in T, there is an extension r_j in T such that for some $n > |r_i|$, we have:

$$(\forall N)(\exists n > N)\left[\frac{|\{i < n \mid \frac{1}{2} < [a_i r_j] < 1\}|}{n} < \frac{1}{4}\right].$$

This guarantees that we can never force the fraction above to converge to the "correct" limit $\frac{1}{2}$, and since G is 2–generic, G must force instead that Ψ^G is not UD–random.

To construct such a sequence $\langle a_i \rangle$, we proceed in stages. At each stage, we have a finite sequence of a_is and add at least one element to it. At stage 0, our sequence is the empty sequence. At each subsequent stage s we deal with p_s. Suppose we have already defined the finite initial segment $\langle a_i \rangle_{i<M}$ of our a-sequence. Look for an r_j such that $p_j \supseteq p_s$ and such that there is an n such that $\frac{|\{i<M \mid \frac{1}{2}<[a_i r_j]<1\}|}{M+n} < \frac{1}{4}$ and r_j contains a 0 at n positions $k_0, k_2 \ldots k_{n-1}$ such that $a_i < 2^{k_\ell}$ for all $i < M$, $\ell \leq n$.

Such an r_j must exist, since we forced every extension of p to compute an $r_i \in T$ with infinitely many zeroes. Of course, we may be in the simplest case, where $n = 0$ suffices, but the treatment is the same.

Set $a_{M+i} = 2^{k_i}$ for all $0 \leq i \leq n-1$. For each such i, the fractional part of $a_{M+i} r_j = 2^{k_i} r_j$ is less than $\frac{1}{2}$ and we have $\frac{|\{i<M+n \mid \frac{1}{2}<[a_i r_j]<1\}|}{M+n} < \frac{1}{4}$ as desired.

Now G must force either the $\Pi_2^{0,Z}$ statement

$$\Theta := (\forall N)(\exists n > N)\left[\frac{|\{i < n \mid \frac{1}{2} < [a_i \Psi^Z] < 1\}|}{n} < \frac{1}{4}\right]$$

or its negation. But G cannot force $\neg\Theta$, so instead G must force Θ, so that $\Psi^G = X$ cannot be UD–random. $\square$

This proof cannot be modified in an obvious way to show that no 1–generic tt-computes a UD–random. Were 2–genericity only necessary to force the totality of Ψ^X, it would be possible, but the other statements in the problem are $\Pi_2^{0,X}$ as well.

4 Future Directions

It seems easier to prove negative results about UD–randoms than positive results. In part, this is due to the need to quantify over all intervals; however, this requirement has been minimized by Fact 1.2. Whether this fact has practical significance remains to be seen. However, the main difficulty in relating UD–randomness to other randomness notions is that UD–randomness is intrinsically a notion about tailsets: when we multiply a real by an integer, an initial segment of the real becomes irrelevant. In contrast, every other common randomness notion is defined in terms of initial segments: those that make up the components of a test or serve as inputs for the martingale or Kolmogorov complexity function.

Question 4.1 Is there a test, martingale, or string (e.g. Kolmogorov) complexity characterization of UD–randomness?

Question 4.2 Do the UD–random reals that are weakly 1–random coincide with the Schnorr random reals?

Question 4.3 Do the UD–random reals coincide with the Schnorr random reals in the hyperimmune-free degrees?

From the comment immediately preceding the questions, it is difficult to see how Question 4.1 could have a positive answer—at least, it is difficult to see how such a characterization could resemble that of other randomness notions in any meaningful way. The authors suspect that the answers to the second two questions are negative as well but have weaker grounds for saying so.

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Department of Mathematics, Mailcode 4408, Southern Illinois University, 1245 Lincoln Drive, Carbondale, IL 62901, USA

Department of Mathematics, Room 306, Roosevelt Hall, Hofstra University, Hempstead, NY 11549-0114, USA

wcalvert@siu.edu, johanna.n.franklin@hofstra.edu

http://lagrange.math.siu.edu/Calvert/,
http://people.hofstra.edu/Johanna_N_Franklin/

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