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Hyperfinite measure-preserving actions of countable groups and their model theory

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Abstract: We give a shorter proof of a theorem of G. Elek stating that two hyperfinite measure-preserving actions of a countable group on standard probability spaces are approximately conjugate if and only if they have the same invariant random subgroup.

We then use this theorem to study model theory of hyperfinite measure-preserving actions of countable groups on probability spaces. This work generalizes the model-theoretic study of automorphisms of probability spaces conducted by I. Ben Yaacov, A. Berenstein, C. W. Henson and A. Usvyatsov.

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1 Introduction

Classical ergodic theory consists of the study of probability measure-preserving (pmp in short) transformations of a probability space. A *pmp transformation* T of a probability space (X, μ) is a bimeasurable permutation of X such that for all measurable subsets A of X, $\mu(T^{-1}A) = \mu(A)$. It is called *ergodic* if any T-invariant subset of X is either null or conull, and it is called *aperiodic* if almost every T-orbit is infinite. In the case of a single transformation T of an atomless probability space, it is well-known that ergodicity implies aperiodicity. For now, we restrict ourselves to *standard probability spaces*, that is probability spaces that are isomorphic to the interval $[0, 1]$ equipped with the Lebesgue measure.

Two pmp transformations T and T' are said to be *conjugate*, or sometimes *isomorphic*, if there is a third pmp transformation S such that up to a null set, $T' = STS^{-1}$. One of the main goals of ergodic theory is to understand the conjugacy relation on pmp transformations, particularly on the set of ergodic pmp transformations. Conjugacy

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is completely understood in some specific cases, for example, entropy is a complete invariant of conjugacy for Bernoulli shifts (Ornstein [17]) and spectrum is a complete invariant of conjugacy for compact transformations. However, in general, conjugacy is a very complicated relation as shown by Foreman and Weiss [10] and Foreman, Rudolph and Weiss [9].

In this paper we study the simpler relation of approximate conjugacy. Two pmp transformations T and T' of (X,μ) are said to be *approximately conjugate* if for all $\varepsilon > 0$ there is a third pmp transformation S of (X,μ) such that $T' = STS^{-1}$ up to a set of measure at most ε. It is a well-known consequence of the Rokhlin Lemma that any two aperiodic pmp transformations of standard probability spaces are approximately conjugate (see Kechris [15, Theorem 2.4]). We thus focus on understanding the approximate conjugacy relation for general pmp actions of countable discrete groups rather than single pmp transformations, which correspond to $\mathbb{Z}$-actions.

A *pmp action* of a countable group Γ on a probability space (X,μ) is an action of Γ on X by pmp transformations. For a pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\gamma \in \Gamma$, we let γ^α denote the pmp transformation associated to γ in the action α. Two pmp actions α and β of a countable group Γ are *conjugate* if there is a pmp transformation S such that $S^{-1}\gamma^\alpha S = \gamma^\beta$ for all $\gamma \in \Gamma$. We say that α is a *factor* of β, denoted by $\alpha \sqsubseteq \beta$ if there is a measure-preserving map $S : X \to X$ such that $\gamma^\alpha S = S\gamma^\beta$ for every $\gamma \in \Gamma$.

We say that α and β are *approximately conjugate* if for every finite $F \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a pmp transformation S of X such that

$$\mu\left(\{x \in X : \exists \gamma \in F,\ \gamma^\beta x \neq S\gamma^\alpha S^{-1}x\}\right) < \varepsilon.$$

This notion of approximate conjugacy comes from the study of the spaces $\mathrm{Aut}(X,\mu)$ and $A(\Gamma, X, \mu)$ of pmp transformations of (X,μ) and of pmp actions of Γ on (X,μ), respectively.

The space $\mathrm{Aut}(X,\mu)$ can be equipped with two topologies: the weak and the uniform topology (see [15] for definitions). Two pmp transformations T and S are called *weakly equivalent* if $\overline{[T]}^w = \overline{[S]}^w$, where $[T]$ is the conjugacy class of T, and $\overline{A}^w$ denotes the closure of A in the weak topology. Then, the space of actions can be seen as a closed subspace of $\mathrm{Aut}(X,\mu)^\Gamma$ equipped with either product topology, and this induces two topologies on $A(\Gamma, X, \mu)$, that we respectively call again the weak and the uniform topology. In the same fashion as for transformations, we say that two actions α and β are weakly equivalent if $\overline{[\alpha]}^w = \overline{[\beta]}^w$.

Now approximate conjugacy is the uniform counterpart of weak equivalence, that is, two pmp actions α and β are approximately conjugate if and only if $\overline{[\alpha]}^u = \overline{[\beta]}^u$, where $\overline{A}^u$ is the uniform closure of A. The study of approximate conjugacy in the present paper was mostly motivated by similar results obtained for weak equivalence by R. Tucker-Drob in [19].

The first obstacle to approximate conjugacy is freeness : a pmp action of Γ is *free* if the set of fixed points of any nontrivial element of Γ is null. For $\mathbb{Z}$-actions, freeness corresponds to aperiodicity. It is easy to see that approximate conjugacy preserves the freeness of the actions, and that the trivial action is only approximately conjugate with itself.

In fact, we have a better result. For a pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$, the pushforward of the measure μ by the stabilizer application $x \in X \mapsto \mathrm{Stab}^\alpha(x)$ gives a measure θ_α on the space of subgroups of Γ. We call this measure the Invariant Random Subgroup (IRS for short, see Abért, Glasner and Virág [1]) of the action α. Then it is not hard to see that the IRS is an invariant of approximate conjugacy. Moreover, free actions correspond to the case where the IRS is the Dirac measure on the trivial subgroup $\delta_{\{e\}}$ and the trivial action corresponds to the case where the IRS is δ_Γ.

In this paper we work with hyperfinite actions, which are defined as follows:

Definition A pmp action $\Gamma \curvearrowright (X,\mu)$ is said to be *hyperfinite* if for any finite subset S of Γ and any $\varepsilon > 0$, there exists a finite group G acting in a measure-preserving way on (X,μ) such that

$$\mu\left(\{x \in X : S \cdot x \subseteq G \cdot x\}\right) > 1 - \varepsilon.$$

It is a theorem of Ornstein and Weiss [18] that pmp actions of amenable groups are hyperfinite.

In general, we have the following implications:

$$\text{approximate conjugacy} \Longrightarrow \text{weak equivalence} \Longrightarrow \text{same IRS}.$$

In the most general context, the IRS of an action is not a complete invariant of approximate conjugacy. However, Elek proved that when restricted to hyperfinite actions, it is:

Theorem A (Elek [8, Theorem 9]) Let α and β be two pmp hyperfinite actions of a group Γ on a standard probability space such that $\theta_\alpha = \theta_\beta$. Then α and β are approximately conjugate.

This theorem thus generalizes the consequence [15, Theorem 2.4] of the Rokhlin Lemma, which can be obtained by taking $\Gamma = \mathbb{Z}$ and $\theta_\alpha = \theta_\beta = \delta_{\{e\}}$.

In this paper, we give a shorter proof of this theorem, first by considering the critical case of actions which are factors one of another and then using a confluence argument to conclude in the general case. Moreover, when one of the actions is a factor of the other, we add a slight improvement to the theorem by requiring that the pmp transformations witnessing approximate conjugacy stabilize some measurable sets. This stronger version of the theorem will be used for the model theoretic study of pmp actions, which is the main topic of the present paper.

The formalism of continuous model theory that we use was developed by I. Ben Yaacov and A. Usvyatsov.

While classical model theory is concerned with algebraic theories such as discrete groups, algebraically closed or real closed fields, its continuous counterpart allows the study of metric structures. In recent years, continuous model theory has been used to study theories such as metrics spaces, Banach spaces, Hilbert spaces and measure algebras. More precisely, a particular attention was given to the study of formulas involving automorphisms of the latter theories.

In the present paper we are interested in the model theory of a group action on a probability space, in other words, we look at formulas involving finite subsets of automorphisms of a probability space (X,μ) from a given subgroup of the group of automorphisms of (X,μ). However, probability spaces do not admit a model theoretic treatment as such, where the elements of a structure are the points in probability spaces.

In order to solve this issue, we consider as structures not the probability spaces themselves but their associated measure algebra. For a probability space (X,Σ,μ), its associated measure algebra $\mathrm{MAlg}(X,\mu)$ is the quotient set $\Sigma/\mathcal{N}$ where $\mathcal{N}$ denotes the σ-ideal of null sets. It inherits the Boolean operations of Σ and is endowed with a natural metric $d_\mu(\pi(A),\pi(B)) := \mu(A \triangle B)$, where π is the quotient map.

Moreover, the correspondence between probability spaces and measure algebras is functorial, so that a pmp action on a probability space induces an action by automorphisms on its measure algebra.

Following the latter remarks, we study the model theory of atomless measure algebras with a countable group Γ acting by automorphisms. This work follows the one in Ben Yaacov, Berenstein, Henson and Usvyatsov [3, Section 18] about free actions of $\mathbb{Z}$ and the more general case of free actions of amenable groups treated by Berenstein and Henson in an unpublished paper.

Without loss of generality, we restrict our study to actions of the free group over an infinite countable subset, F_∞, as any action of a countable group can be seen as an action of F_∞. Then one can see that the equivalence relation of elementary equivalence is weaker than approximate conjugacy but stronger than weak equivalence. This result highlights the link between model theory and the equivalence relations usually studied in ergodic theory.

For any IRS θ on F_∞, we define a theory $\mathfrak{A}_\theta$ axiomatizing pmp actions having IRS θ. By a result of Elek ([8, Theorem 2]), the hyperfiniteness of an action is determined by its IRS. We thus call an IRS θ hyperfinite if actions having IRS θ are hyperfinite.

By Theorem A, in the context of hyperfinite actions, having the same IRS is equivalent to being elementarily equivalent. We prove:

Theorem B If θ is a hyperfinite IRS, then the theory $\mathfrak{A}_\theta$ is complete and model complete.

However, unlike in [3, Section 18] these theories do not admit quantifier elimination in general. We nevertheless prove in Theorem 3.31 that there is a reasonable expansion of the theory which eliminates quantifier, and we then use this to prove

Theorem C If θ is a hyperfinite IRS, then the theory $\mathfrak{A}_\theta$ is stable and the stable independence relation given by non dividing admits a natural characterization in terms of the classical probabilistic independence of events (in a sense described in Definition 3.37).

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2 Generalization of the Rokhlin Lemma

2.1 Graphings

Definition 2.1 A *graph* G is a pair $(\mathrm{V}(G), \mathrm{E}(G))$ where $\mathrm{V}(G)$ is a set and $\mathrm{E}(G)$ is an irreflexive and symmetric binary relation on $\mathrm{V}(G)$. Elements of $\mathrm{V}(G)$ are called *vertices* of G and elements of $\mathrm{E}(G)$ are called *edges* of G.

For G a graph, for each $v \in \mathrm{V}(G)$ we let $\deg_G(v) = |\{u \in \mathrm{V}(G) : (v, u) \in \mathrm{E}(G)\}|$, and we call $\sup_{v \in \mathrm{V}(G)} \deg_G(v) \in \mathbb{N} \cup \{\infty\}$ the degree bound of G.

Definition 2.2 An *isomorphism between the graphs G and H* is a bijective map $f\colon \mathrm{V}(G) \to \mathrm{V}(H)$ such that

$$\forall x, y \in \mathrm{V}(G), \quad (x, y) \in \mathrm{E}(G) \iff (f(x), f(y)) \in \mathrm{E}(H).$$

Definition 2.3 Let G be a graph, $A \subseteq \mathrm{V}(G)$ and $B \subseteq \mathrm{E}(G)$. Then we define :

- $\mathrm{V}^G_{\mathrm{inc}}(B) = \{v \in \mathrm{V}(G) : \exists u \in \mathrm{V}(G),\ (u, v) \in B \vee (v, u) \in B\}$ the set of *vertices incident to B*.
- $\mathrm{E}^G_{\mathrm{inc}}(A) = \{(a, v) \in \mathrm{E}(G) : a \in A\}$ the set of *edges incident to A*.

We will write $\mathrm{V}_{\mathrm{inc}}(B)$ and $\mathrm{E}_{\mathrm{inc}}(A)$ when the context makes clear which graph G is considered.

Definition 2.4 Let G be a graph. A *subgraph* of G is a graph H such that $\mathrm{V}(H) = \mathrm{V}(G)$ and $\mathrm{E}(H) \subseteq \mathrm{E}(G)$. In this case, we write $H \subseteq G$.

If $\mathrm{V} \subseteq \mathrm{V}(G)$, the *subgraph of G induced by V* is the graph $(\mathrm{V}(G), \mathrm{E}(G) \cap V^2)$. Nevertheless, in many cases it will be convenient to identify the induced graph on V and the graph $(V, \mathrm{E}(G) \cap V^2)$ and therefore see the induced graph on V as a graph on the set of vertices V.

We write $G \setminus \mathrm{V}$ for the subgraph of G induced by $\mathrm{V}(G) \setminus \mathrm{V}$.

In general, we write $G \simeq H$ to indicate that G and H are isomorphic.

Definition 2.5 A *standard Borel space* is a measurable space isomorphic to $[0, 1]$ equipped with its Borel σ-algebra. We call Borel the maps between two standard Borel spaces which are measurable.

Let us give some notations regarding probability spaces :

- If X is a measurable space, we denote by $\mathfrak{P}(X)$ the set of probability measures on X.
- If (X, μ) is a probability space and P is a property, we write $\forall^* x \in X\ P(x)$ for $\mu(\{x \in X : P(x)\}) = 1$ and $\exists^* x \in X\ P(x)$ for $\mu(\{x \in X : P(x)\}) > 0$.
- If (X, μ) is a probability space, Y is a measurable space and $T : X \to Y$ is a measurable map, we write $T_*\mu$ for the pushforward of μ by T, that is the measure in $\mathfrak{P}(Y)$ defined by $T_*\mu(A) = \mu(T^{-1}(A))$ for any Borel subset $A \subseteq Y$.

Definition 2.6 Let X be a standard Borel space and $\mathcal{R}$ be a Borel (as a subset of the measurable space $X \times X$) equivalence relation on X with countable classes. We let $[\mathcal{R}]$ be the group of Borel automorphisms of X whose graphs are contained in $\mathcal{R}$. We say that a Borel probability measure μ on X is *$\mathcal{R}$-invariant* if every element of $[\mathcal{R}]$ preserves the measure μ, namely, $\forall T \in [\mathcal{R}],\ T_*\mu = \mu$.

Example 2.7 Given a countable group Γ acting on (X, μ) in a Borel manner, one can form the Borel equivalence relation $\mathcal{R}_\Gamma = \{(x, \gamma \cdot x) \colon \gamma \in \Gamma, x \in X\}$ whose classes are exactly the Γ-orbits. It is a well known fact that the Γ-action is pmp (meaning that for all $\gamma \in \Gamma$, we have $\gamma_*\mu = \mu$) if and only if μ is $\mathcal{R}_\Gamma$-invariant (see Kechris and Miller [16, Proposition 2.1]). We will often use both implications without explicit mention.

Remark 2.8 It is a straightforward consequence of the definition that if $\mathcal{R}$ preserves μ, then so does every Borel subequivalence relation of $\mathcal{R}$. In particular, every Borel action of a countable group whose orbits are contained in the equivalence classes of an equivalence relation which preserves μ must be a pmp action.

Proposition 2.9 ([16, Section 8]) *With the same notations as above, for any $\mu \in \mathfrak{P}(X)$, we can define two measures μ_l and μ_r on $\mathcal{R}$ by*

- *for all non-negative Borel $f \colon \mathcal{R} \to [0, \infty]$, $\int_{\mathcal{R}} f \, d\mu_l = \int_X \sum_{y \in [x]_{\mathcal{R}}} f(x, y) \, d\mu(x)$,*
- *for all non-negative Borel $f \colon \mathcal{R} \to [0, \infty]$, $\int_{\mathcal{R}} f \, d\mu_r = \int_X \sum_{y \in [x]_{\mathcal{R}}} f(y, x) \, d\mu(x)$,*

where $[x]_{\mathcal{R}}$ denotes the equivalence class of x for $\mathcal{R}$. Then $\mu_l = \mu_r$ if and only if μ is $\mathcal{R}$-invariant.

Definition 2.10 Let $\mathcal{G}$ be a Borel graph on a standard probability space (X, μ) which has countable connected components. Then the equivalence relation $\mathcal{R}_\mathcal{G}$ induced by $\mathcal{G}$ is the equivalence relation on (X, μ) whose classes are the connected components of $\mathcal{G}$. By the Lusin–Novikov theorem, $\mathcal{R}_\mathcal{G}$ is a Borel equivalence relation. We say that $\mathcal{G}$ is a *graphing* when μ is $\mathcal{R}_\mathcal{G}$-invariant.

We can define a measure on the set of edges of a graphing by:

Definition 2.11 Let $\mathcal{G}(X, \mu)$ be a graphing and $Z \subseteq \mathrm{E}(\mathcal{G})$ be a Borel set. The *edge measure* of the set Z is defined by $\mu_\mathrm{E}(Z) := \mu_l(Z) = \mu_r(Z)$, where μ_l and μ_r are defined with respect to the Borel equivalence relation $\mathcal{R}_\mathcal{G}$.

For a graphing of degree bound d, the edge measure of a set of edges is bounded by the measure of the vertices incident to this set. Namely, for all Borel $Z \subseteq \mathrm{E}(\mathcal{G})$ we have

$$\frac{1}{2}\mu(\mathrm{V}_{\mathrm{inc}}(Z)) \leq \mu_\mathrm{E}(Z) \leq d\mu(\mathrm{V}_{\mathrm{inc}}(Z)).$$

2.2 Classical Rokhlin Lemma

A measure-preserving transformation is called *aperiodic* if almost all its orbits are infinite.

The Rokhlin Lemma states that if T is an aperiodic measure-preserving transformation of a standard probability space (X, μ), then for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there is a Borel subset $A \subseteq X$ such that the sets $A, TA, \ldots, T^{n-1}A$ are pairwise disjoint and

$$\mu\left(\bigsqcup_{i=0}^{n-1} T^i A\right) > 1 - \varepsilon.$$

What we present in this paper is not a generalization of the Rokhlin Lemma itself but rather of one of its important and well-known consequences:

Corollary 2.12 (Uniform Approximation Theorem, [15, Theorem 2.2]) *Any two aperiodic measure-preserving transformations τ_1 and τ_2 on standard probability spaces (X, μ) and (Y, ν) are approximately conjugate.*

An aperiodic measure-preserving transformation can be seen as a free action of $\mathbb{Z}$. The goal of this section is to generalize the latter Corollary to hyperfinite actions of a countable group which have a given IRS (i.e. Invariant Random Subgroup, defined in Subsection 2.4).

2.3 Hyperfiniteness

The key point in the proof of Uniform Approximation Theorem 2.12 is that the dynamics of an aperiodic automorphism of a standard probability space (X, μ) are understood on arbitrarily large proper subsets of X. In this section we define the notion of hyperfiniteness of a pmp action, which allows one to make this idea work in a much more general context.

Definition 2.13 (See "approximately finite group" in Dye [7]) A pmp action $\Gamma \curvearrowright (X, \mu)$ is said to be *hyperfinite* if for every finite $S \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a finite group G acting in a measure-preserving way on (X, μ) such that

$$\mu\left(\{x \in X : S \cdot x \subseteq G \cdot x\}\right) > 1 - \varepsilon.$$

What we are mostly interested in is the characterization of hyperfiniteness for graphings.

Definition 2.14 Let $\mathcal{G}(X,\mu)$ be a graphing. $\mathcal{G}$ is called *hyperfinite* if for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ and a Borel set $Z \subseteq \mathrm{E}(\mathcal{G})$ such that $\mu_{\mathrm{E}}(Z) < \varepsilon$ and the subgraphing $\mathcal{H} = \mathcal{G} \setminus Z$ has components of size at most M.

Definition 2.15 Let F be a finite set. An *F-colored graphing* on a standard probability space (X,μ) is a graphing $\mathcal{G}(X,\mu)$ endowed with a Borel map $\varphi_{\mathcal{G}}\colon \mathrm{E}(\mathcal{G}) \to F$. For $(x,y) \in \mathrm{E}(\mathcal{G})$, we call $\varphi_{\mathcal{G}}(x,y)$ the color of (x,y).

Additionally, for $c \in F$, we write $\mathrm{E}^c(\mathcal{G})$ for the set of edges colored by c, namely $\varphi_{\mathcal{G}}^{-1}(c)$.

We will simply write $\mathcal{G}$ and consider the color implicitly when dealing with colored graphings.

Definition 2.16 Let $\mathcal{G}(X,\mu)$ and $\mathcal{G}'(Y,\nu)$ be two F-colored graphings. A *colored graphing factor map* $\pi\colon Y \to X$ is a pmp map such that for almost all $y \in Y$, $\pi\restriction_{[y]_{\mathcal{G}'}}$ is an isomorphism of F- colored graphs.

We say that $\mathcal{G}$ is a colored factor of $\mathcal{G}'$, and we write $\mathcal{G} \underset{c}{\sqsubseteq} \mathcal{G}'$, if there is a colored graphing factor map $\pi\colon Y \to X$.

Let Γ be a group and S be a finite subset of Γ. Let us consider a measure-preserving action $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$. We define a graph $\mathcal{G}_{\alpha,S}$ on (X,μ) by $(x,y) \in \mathrm{E}(\mathcal{G}_{\alpha,S})$ if and only if there is an $s \in S$ such that $y = sx$. Remark 2.8 ensures that $\mathcal{G}_{\alpha,S}$ is a graphing.

We then color the edges of $\mathcal{G}_{\alpha,S}$ by letting the color of the edge (x,y) be $\{s \in S : y = sx\}$. This makes $\mathcal{G}_{\alpha,S}$ a $\mathcal{P}(S)$-colored graphing. We call it the *Schreier graph* of the action α relative to S.

Lemma 2.17 *Let Γ be a countable group and let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ be a pmp action. Then α is hyperfinite if and only if for every finite $S \subseteq \Gamma$, $\mathcal{G}_{\alpha,S}$ is hyperfinite.*

Proof Suppose α is hyperfinite and let $S \subseteq \Gamma$ be finite and $\varepsilon > 0$.

By hyperfiniteness, there exists a finite group G along with a pmp action $G \curvearrowright (X,\mu)$ such that $\mu(\{x \in X : S \cdot x \subseteq G \cdot x\}) > 1 - \varepsilon$. In particular, when restricted to the set $\{x \in X : S \cdot x \subseteq G \cdot x\}$, the Schreier graph $\mathcal{G}_{\alpha,S}$ has finite components of size less than $|G|$.

For the converse, suppose that for any $S \subseteq \Gamma$ finite, the graphing $\mathcal{G}_{\alpha,S}$ is hyperfinite.

Let $S \subseteq \Gamma$ be finite and let $\varepsilon > 0$. Then there exist $Z \subseteq \mathrm{E}(\mathcal{G}_{\alpha,S})$ Borel and $M \in \mathbb{N}$ such that $\mu_{\mathrm{E}}(Z) < \frac{\varepsilon}{2}$ and $\mathcal{G}_{\alpha,S} \setminus Z$ has components of size at most M.

We define a pmp action of $\prod_{n \leq M} \mathbb{Z}/n\mathbb{Z}$ on (X, μ) as follows:

Since (X, μ) is a standard probability space, there is a Borel linear ordering $<$ of X. This induces, for $n \leq M$, an action of $\mathbb{Z}/n\mathbb{Z}$ on X that shifts $\mathcal{G}_{\alpha,S} \setminus Z$-components of size exactly n according to the order $<$ and acts trivially on other components. By Remark 2.8, this action preserves μ because its orbit equivalence relation is a subset of the orbit equivalence relation of α.

It follows that $\prod_{n \leq M} \mathbb{Z}/n\mathbb{Z}$ acts as a product on $X \setminus Z$ in a pmp way.

One can easily check that for $x \notin \mathrm{V}_{\mathrm{inc}}(Z)$, $S \cdot x$ is exactly the set of neighbors of x in $\mathcal{G}_{\alpha,S} \setminus Z$, and thus it is contained in $[x]_{\mathcal{G}_{\alpha,S} \setminus Z} = \left(\prod_{n \leq M} \mathbb{Z}/n\mathbb{Z}\right) \cdot x$. Moreover, $\mu(\mathrm{V}_{\mathrm{inc}}(Z)) \leq 2\mu_{\mathrm{E}}(Z) < \varepsilon$ so we conclude that α is hyperfinite. □

2.4 Invariant Random Subgroups

Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ be a measure-preserving action of the countable group Γ on a standard probability space. With this action we can associate a probability measure on the Polish space of subgroups of Γ as follows.

Consider the compact Polish space $\{0, 1\}^{\Gamma}$. We let $\mathrm{Sub}(\Gamma)$ be the closed subset of $\{0, 1\}^{\Gamma}$ consisting of the subgroups of Γ. Then $\mathrm{Sub}(\Gamma)$ is a compact Polish space, and we make it a measurable space by endowing it with its Borel σ-algebra. We have a natural Borel map $\mathrm{Stab}^{\alpha} \colon X \to \mathrm{Sub}(\Gamma)$ defined by $x \mapsto \mathrm{Stab}^{\alpha}(x) = \{g \in \Gamma : g^{\alpha}(x) = x\}$ and the pushforward construction gives us a probability measure $\mathrm{Stab}^{\alpha}_{*}\mu \in \mathfrak{P}(\mathrm{Sub}(\Gamma))$ that we call the Invariant Random Subgroup (IRS in short) of α and denote by θ_{α}. Moreover, Γ acts on $\mathrm{Sub}(\Gamma)$ by conjugacy and the well known formula $\mathrm{Stab}^{\alpha}(gx) = g\mathrm{Stab}^{\alpha}(x)g^{-1}$ implies that the map Stab^{α} is equivariant. Therefore, θ_{α} is a Γ-invariant measure on $\mathrm{Sub}(\Gamma)$. We thus define the general notion of an *IRS* on Γ to be a Borel probability measure on $\mathrm{Sub}(\Gamma)$ invariant for the action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by conjugacy.

Abért, Glasner and Virág proved in [1, Proposition 13] that any IRS can be obtained as the IRS associated to a pmp action. Moreover, Elek proved in [8, Theorem 2] that two pmp actions of a countable group Γ with the same IRS are either both hyperfinite or both non-hyperfinite. We can thus express hyperfiniteness as a property of the IRS itself:

Definition 2.18 Let Γ be a countable group. An IRS θ on Γ is called *hyperfinite* if one of the following two equivalent statements is satisfied :

(1) There exists a hyperfinite pmp action which has IRS θ.

(2) Every pmp action which has IRS θ is hyperfinite.

Definition 2.19 Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (Y,\nu)$. An *action factor map* $\pi\colon Y \to X$ is a measure-preserving map such that $\forall^* y \in Y\ \forall \gamma \in \Gamma,\ \pi(\gamma^\beta y) = \gamma^\alpha \pi(y)$.

We say that α *is a factor of* β, and we write $\alpha \sqsubseteq \beta$, if there exists an action factor map $\pi\colon Y \to X$.

Lemma 2.20 *Let α, β be two actions of a countable group Γ on standard probability spaces (X,μ) and (Y,ν). Suppose that there is an action factor map $\pi\colon Y \to X$ for α and β and that $\theta_\alpha = \theta_\beta$. Then $\forall^* y \in Y,\ \mathrm{Stab}^\alpha(\pi(y)) = \mathrm{Stab}^\beta(y)$.*

Proof For $\gamma \in \Gamma$, let $\mathrm{N}_\gamma = \{\Lambda \in \mathrm{Sub}(\Gamma) : \gamma \in \Lambda\}$. Then $(\mathrm{N}_\gamma)_{\gamma\in\Gamma}$ is a subbasis of the topology of $\mathrm{Sub}(\Gamma)$ consisting of clopen sets.

By the definition of action factor map, we have $\forall^* y\ \mathrm{Stab}^\beta(y) \subseteq \mathrm{Stab}^\alpha(\pi(y))$. Suppose now that $\exists^* y\ \mathrm{Stab}^\beta(y) \subsetneq \mathrm{Stab}^\alpha(\pi(y))$.

By countability of Γ, $\exists \gamma \in \Gamma\ \exists^* y,\ \gamma \in \mathrm{Stab}^\alpha(\pi(y)) \setminus \mathrm{Stab}^\beta(y)$, thus

$$\begin{aligned}
\theta_\beta(\mathrm{N}_\gamma) &= \mathrm{Stab}^\beta_*\nu(\mathrm{N}_\gamma)\\
&< (\mathrm{Stab}^\alpha \circ \pi)_*\nu(\mathrm{N}_\gamma)\\
&= \mathrm{Stab}^\alpha_*(\pi_*\nu)(\mathrm{N}_\gamma)\\
&= \mathrm{Stab}^\alpha_*\mu(\mathrm{N}_\gamma)\\
&= \theta_\alpha(\mathrm{N}_\gamma),
\end{aligned}$$

a contradiction to our hypothesis $\theta_\alpha = \theta_\beta$. □

Corollary 2.21 *Let α, β be actions of a countable group Γ on standard probability spaces (X,μ) and (Y,ν) such that $\alpha \sqsubseteq \beta$ and $\theta_\alpha = \theta_\beta$, and let $S \subseteq \Gamma$ be finite. Then we have $\mathcal{G}_{\alpha,S} \underset{c}{\sqsubseteq} \mathcal{G}_{\beta,S}$ as $\mathcal{P}(S)$-colored graphings.*

Proof Applying Lemma 2.20 to an action factor map $\pi\colon Y \to X$ gives us that for almost every $y \in Y$, $\pi \restriction_{\Gamma\cdot y}$ is a Γ-equivariant bijection $\Gamma \cdot y \to \Gamma \cdot \pi(y)$ and so it is an isomorphism of $\mathcal{P}(S)$-colored graphs. It follows that π is a graphing factor map. □

We end this section with a crucial and well-known observation: every IRS is determined by the measure it assigns to clopen sets of the form $N_{F,\emptyset}$, where for finite subsets $F, G \subseteq \Gamma$ we let

$$\mathrm{N}_{F,G} = \{\Lambda \leq \Gamma : F \subseteq \Lambda, G \cap \Lambda = \varnothing\}.$$

Lemma 2.22 *Let θ_1 and θ_2 be two probability measures on* $\mathrm{Sub}(\Gamma)$ *such that for all* $F \subseteq \Gamma$, *we have*

$$\theta_1(\mathrm{N}_{F,\emptyset}) = \theta_2(\mathrm{N}_{F,\emptyset})$$

Then $\theta_1 = \theta_2$.

Proof First observe that since Γ is countable, $\mathrm{Sub}(\Gamma)$ is second-countable and hence the Borel σ-algebra of $\mathrm{Sub}(\Gamma)$ is generated by any basis of its topology. Now observe that sets of the form $\mathrm{N}_{F,G}$ where F and G are finite, as defined right before the lemma, do form such a basis, hence they generate the Borel σ-algebra. Furthermore, every $\mathrm{N}_{F,G}$ is contained in the algebra generated by sets of the form $\mathrm{N}_{F',\emptyset}$, and since the latter also form a π-system, any Borel probability measure on $\mathrm{Sub}(F_\infty)$ is determined by its values on sets of the form $\mathrm{N}_{F',\emptyset}$ where F' ranges over finite subsets of Γ by the π-λ theorem (see Cohn [6, Corollary 1.6.3]). □

Remark 2.23 The above result can also be proven more concretely by appealing to the inclusion-exclusion principle so as to recover the measure $\mu(\mathrm{N}_{F,G})$ from the measures of sets of the form $\mathrm{N}_{F',\emptyset}$, see Tucker-Drob [19, end of the proof of Theorem 5.2].

2.5 The proof of Theorem A

2.5.1 The preliminary case of factors

We begin with the case where one of the actions is a factor of the other. In fact, we prove a stronger version involving the stability of Borel sets.

Definition 2.24 Let F_1, F_2 be two finite sets. An (F_1, F_2)*-bicolored graphing* on a standard probability space (X, μ) is a graphing $\mathcal{G}(X, \mu)$ endowed with two Borel maps $\varphi_{\mathcal{G}} \colon \mathrm{E}(\mathcal{G}) \to F_1$ and $\psi_{\mathcal{G}} \colon X \to F_2$. We call $\psi_{\mathcal{G}}(x)$ the vertex-color of x and $\varphi_{\mathcal{G}}(x, y)$ the edge-color of (x, y).

Definition 2.25 Let $\mathcal{G}(X, \mu)$ and $\mathcal{G}'(Y, \nu)$ be two (F_1, F_2)-bicolored graphings. A *bicolored graphing factor map* $\pi \colon Y \to X$ is an F_1-colored graphing factor map such that $\psi_{\mathcal{G}} \circ \pi = \psi_{\mathcal{G}'}$.

We say that $\mathcal{G}$ is a bicolored factor of $\mathcal{G}'$, and we write $\mathcal{G} \underset{bic}{\sqsubseteq} \mathcal{G}'$, if there is a bicolored factor map $\pi \colon Y \to X$.

Lemma 2.26 *Let $\mathcal{G}(X,\mu)$ and $\mathcal{G}'(Y,\nu)$ be hyperfinite (F_1,F_2)-bicolored graphings of degree bound at most d such that $\mathcal{G}(X,\mu) \underset{bic}{\sqsubseteq} \mathcal{G}'(Y,\nu)$. Then for any $\varepsilon > 0$ there exists a pmp bijection $\rho\colon X \to Y$ such that $\psi_{\mathcal{G}} = \psi_{\mathcal{G}'} \circ \rho$ and*

$$\mu_{\mathrm{E}}\left(\bigcup_{c\in F_1} \rho^{-1}\left(\mathrm{E}^c(\mathcal{G}')\right) \triangle \mathrm{E}^c(\mathcal{G})\right) < \varepsilon.$$

Proof Let π be a bicolored graphing factor map $Y \to X$. First take a Borel set $Z \subseteq \mathrm{E}(\mathcal{G})$ of measure less than $\frac{\varepsilon}{2d}$ and $M \in \mathbb{N}$ such that the graphing $\mathcal{H} = \mathcal{G} \setminus Z$ has components of size at most M. Let $Z' = \pi^{-1}(Z)$ and $\mathcal{H}' = \mathcal{G}' \setminus Z'$. Since π is a graphing factor map, we know that $\mathcal{H}'$ has components of size at most M. Then $\mathcal{H}$ and $\mathcal{H}'$ have a (F_1,F_2)-bicolored graphing structure respectively for the maps $\varphi_{\mathcal{G}} \restriction_{\mathrm{E}(\mathcal{H})}$, $\psi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}'} \restriction_{\mathrm{E}(\mathcal{H}')}$, $\psi_{\mathcal{G}'}$.

Consider the set $\mathcal{G}_M$ of connected (F_1,F_2)-colored graphs of size at most M. We consider the two partitions $X = \bigsqcup_{S\in\mathcal{G}_M} C_S^{\mathcal{H}}$ and $Y = \bigsqcup_{S\in\mathcal{G}_M} C_S^{\mathcal{H}'}$, where $C_S^{\mathcal{H}}$ is defined to be the set of vertices of $\mathcal{H}$ whose component is (F_1,F_2)-colored isomorphic to S. Since π induces (F_1,F_2)- colored graph isomorphisms, we have $C_S^{\mathcal{H}'} = \pi^{-1}(C_S^{\mathcal{H}})$.

In order to define ρ, it suffices to define a measure-preserving bijection $\rho_S\colon C_S^{\mathcal{H}} \to C_S^{\mathcal{H}'}$ preserving bicolored graph structures for each $S \in \mathcal{G}_M$.

Indeed, the union of all these bijections would yield a measure-preserving bijection $\rho\colon X \to Y$ preserving vertex-colors such that $\forall x \in X \setminus \mathrm{V}_{\mathrm{inc}}(Z)$, $\mathrm{B}^{\mathcal{G}}(x,1) = \mathrm{B}^{\mathcal{H}}(x,1) \simeq \mathrm{B}^{\mathcal{H}'}(\rho(x),1) = \mathrm{B}^{\mathcal{G}'}(\rho(x),1)$, where $\mathrm{B}^{G}(v,n)$ denotes the ball of size n centered at v in the graph G. Hence, we would have

$$\mathrm{V}_{\mathrm{inc}}\left(\bigcup_{c\in F_1} \rho^{-1}\left(\mathrm{E}^c(\mathcal{G}')\right) \triangle \mathrm{E}^c(\mathcal{G})\right) \subseteq \mathrm{V}_{\mathrm{inc}}(Z),$$

and so

$$\mu_{\mathrm{E}}\left(\bigcup_{c\in F_1} \rho^{-1}\left(\mathrm{E}^c(\mathcal{G}')\right) \triangle \mathrm{E}^c(\mathcal{G})\right) \leq d\mu(\mathrm{V}_{\mathrm{inc}}(Z)) \leq 2d\mu_{\mathrm{E}}(Z) < \varepsilon.$$

Take $S \in \mathcal{G}_M$ and let us define ρ_S. First we define a partition of $C_S^{\mathcal{H}}$ into Borel transversals $(T_v)_{v\in\mathrm{V}(S)}$ (for $\mathcal{H}$) by induction, such that the elements of T_v occupy the same place in their component for $\mathcal{H}$ as v in S.

Suppose that the $T_{v'}$ are already defined for $v' \in R$ where R is a proper subset of $\mathrm{V}(S)$. Take $v \in \mathrm{V}(S) \setminus R$ incident to R and let $\widetilde{T}_v = \{x \in C_S^{\mathcal{H}} : ([x]_{\mathcal{H}}, x) \simeq_R (S,v)\}$. Here $\simeq_R$

means isomorphic over R, that is there exists an isomorphism $f\colon ([x]_{\mathcal{H}}, x) \to (S, v)$ of colored rooted graphs such that $\forall v' \in R$, $f([x]_{\mathcal{H}} \cap T_{v'}) = \{v'\}$. Now since $\mathcal{H}$ has finite components, chose for T_v any Borel transversal of $\widetilde{T}_v$. Then we let $R' = R \cup \{v\}$ and we iterate the construction.

Again since π is a bicolored graphing factor map, the family $(\pi^{-1}(T_v))_{v \in \mathrm{V}(S)}$ is a partition of $C_S^{\mathcal{H}'}$ into Borel transversals (for $\mathcal{H}'$) such that the elements of $\pi^{-1}(T_v)$ occupy the same place in their component for $\mathcal{H}'$ as v in S. For $v \in \mathrm{V}(S)$, let $t_v\colon C_S^{\mathcal{H}} \to T_v$ be the Borel map that sends any element x to the unique element in $[x]_{\mathcal{H}} \cap T_v$ and $t'_v\colon C_S^{\mathcal{H}'} \to \pi^{-1}(T_v)$ be the Borel map that sends any element y to the unique element in $[y]'_{\mathcal{H}} \cap \pi^{-1}(T_v)$. We may now define ρ_S:

- First choose $v_0 \in S$ and a measure-preserving bijection $\rho_S^{v_0}\colon T_{v_0} \to \pi^{-1}(T_{v_0})$.
- Then for every $v \in S$, $t'_v \circ \rho_S^{v_0} \circ t_{v_0}$ is the only extension of $\rho_S^{v_0}$ to T_v that respects the graph structure of S. Thus define $\rho_S\colon C_S^{\mathcal{H}} \to C_S^{\mathcal{H}'}$ by $\rho_{S \restriction T_v} = t'_v \circ \rho_S^{v_0} \circ t_{v_0}$ for any $v \in \mathrm{V}(S)$.

Now all the maps of the form t_v or t'_v are restrictions of elements of $[\mathcal{R}_{\mathcal{G}}]$ and $[\mathcal{R}'_{\mathcal{G}}]$ respectively, so they are pmp, and therefore ρ_S is pmp. Moreover, as π is a colored graphing factor map, it is clear that for every $x \in C_S^{\mathcal{H}}$, ρ_S induces an isomorphism of colored graphs between $[x]_{\mathcal{H}}$ and $[\rho_S(x)]_{\mathcal{H}'}$. □

Theorem 2.27 (Approximate parametrized conjugacy for factor actions) *Let (X, μ) and (Y, ν) be standard probability spaces and $A_1, \ldots, A_k \subseteq X$, $B_1, \ldots, B_k \subseteq Y$ be Borel subsets. Let Γ be a countable group, θ be a hyperfinite IRS on Γ and $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu), \Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$ be pmp actions of Γ having IRS θ and such that $\alpha \sqsubseteq \beta$ for an action factor map $\pi\colon Y \to X$ such that $\forall i \leq k$, $\pi^{-1}(A_i) = B_i$. Then for $\varepsilon > 0$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, there exists a pmp bijection $\rho\colon X \to Y$ such that $\forall i \leq k$, $\rho(A_i) = B_i$ and*

$$\mu(\{x \in X : \forall i \leq n,\ \rho \circ \gamma_i^{\alpha}(x) = \gamma_i^{\beta} \circ \rho(x)\}) > 1 - \varepsilon.$$

Proof We want to apply Lemma 2.26 to suitable graphings. Let

$$S = \{\gamma_1, \ldots, \gamma_n, \gamma_1^{-1}, \ldots, \gamma_n^{-1}\}$$

and consider the graphings $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$.

For the spaces of colors, we choose $F_1 = \mathcal{P}(S)$ and $F_2 = \mathcal{P}(\{1, \ldots, k\})$. The way we color edges has already been explained; for vertices, simply color a vertex $x \in X$ by $\psi_{\mathcal{G}_{\alpha,S}}(x) = \{i \leq k : x \in A_i\}$ and $y \in Y$ by $\psi_{\mathcal{G}_{\beta,S}}(y) = \{i \leq k : y \in B_i\}$.

First, $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$ are indeed $(\mathcal{P}(S),\mathcal{P}(\{1,\dots,k\}))$-bicolored graphings, and are hyperfinite since α and β are hyperfinite actions.

The next step is to prove that π considered in the statement of the theorem is a bicolored factor map for the $(\mathcal{P}(S),\mathcal{P}(\{1,\dots,k\}))$-bicolored graphings $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$.

- First, π is indeed a pmp map $Y\to X$.
- Then for $y\in Y$, we have
$$\psi_{\mathcal{G}_{\alpha,S}}(\pi(y))=\{i\leq k:\pi(y)\in A_i\}=\{i\leq k:y\in B_i\}=\psi_{\mathcal{G}_{\beta,S}}(y).$$
- Finally, by Corollary 2.21, π is furthermore a colored graphing factor map between the $\mathcal{P}(S)$-colored graphings $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$.

Applying the Claim gives us a pmp bijection $\rho\colon X\to Y$ such that $\psi_{\mathcal{G}_{\alpha,S}}=\psi_{\mathcal{G}_{\beta,S}}\circ\rho$ and
$$\mu_{\mathrm{E}}\left(\bigcup_{c\in\mathcal{P}(S)}\mathrm{E}^c(\mathcal{G}_{\alpha,S})\,\triangle\,\rho^{-1}\left(\mathrm{E}^c(\mathcal{G}_{\beta,S})\right)\right)<\frac{\varepsilon}{2}.$$
But then for $1\leq i\leq k$, $\rho(A_i)=B_i$, and by definitions of $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$ we get
$$\{x\in X:\exists\gamma\in S,\ \rho\circ\gamma^{\alpha}(x)\neq\gamma^{\beta}\circ\rho(x)\}\subseteq\mathrm{V}_{\mathrm{inc}}\left(\bigcup_{c\in\mathcal{P}(S)}\mathrm{E}^c(\mathcal{G}_{\alpha,S})\,\triangle\,\rho^{-1}\left(\mathrm{E}^c(\mathcal{G}_{\beta,S})\right)\right),$$
so its measure is less than $2\cdot\frac{\varepsilon}{2}=\varepsilon$. □

2.5.2 Amalgamation of measure-preserving actions

To conclude the proof of Theorem A, we will use the transitivity of the approximate conjugacy relation and show that for any two pmp actions $\Gamma\overset{\alpha}{\curvearrowright}(X,\mu)$ and $\Gamma\overset{\beta}{\curvearrowright}(Y,\nu)$ of Γ such that $\theta_\alpha=\theta_\beta$, there is a third pmp action $\Gamma\overset{\zeta}{\curvearrowright}(Z,\eta)$ such that $\theta_\eta=\theta_\alpha=\theta_\beta$ and both α and β are factors of ζ.

We recall the definition of the relative independent joining following the presentation in [13].

Proposition 2.28 (Disintegration theorem,[13, A.7]) *Let X,Y be standard probability spaces, $\mu\in\mathfrak{P}(Y)$ and $\pi\colon Y\to X$ be a measurable map. We let $\nu=\pi_*\mu$. Then there is a ν-a.e. uniquely determined family of probability measures $(\mu_x)_{x\in X}\in\mathfrak{P}(Y)^X$ such that:*

(1) *For each Borel* $B \subseteq Y$, *the map* $x \mapsto \mu_x(B)$ *is measurable.*

(2) *For* ν*-a.e.* $x \in X$, μ_x *is concentrated on the fiber* $\pi^{-1}(x)$.

(3) *For every Borel map* $f\colon Y \to [0,\infty]$, $\int_Y f(y)\, d\mu(y) = \int_X \int_Y f(y)\, d\mu_x(y)\, d\nu(x)$.

We then write $\mu = \int_X \mu_x \, d\nu$.

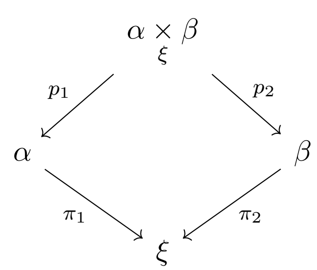
Definition 2.29 (Glasner [13, Section 6.1]) Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (X',\mu')$ be pmp actions on standard probability spaces, and let $\Gamma \overset{\xi}{\curvearrowright} (Y,\nu)$ be an action on a standard probability space common factor of α and β for respective action factor maps $\pi\colon X \to Y$ and $\pi'\colon X' \to Y$.

We can disintegrate μ and μ' with respect to ν using the Borel maps π and π' to get $\mu = \int_Y \mu_y \, d\nu$ and $\mu' = \int_Y \mu'_y \, d\nu$.

Consider $Z := X \times Y$ and $\eta \in \mathfrak{P}(Z)$ defined by $\eta = \int_Y \mu_y \times \mu'_y \, d\nu$.

The pmp action $\Gamma \overset{\alpha\times\beta}{\curvearrowright} (Z,\eta)$ is called the *independent joining of* α *and* β *over* ξ and is denoted by $\alpha \underset{\xi}{\times} \beta$.

The action $\alpha \underset{\xi}{\times} \beta$ is indeed a *joining* of α and β over ξ, meaning that both α and β are factors of their independent joining over ξ, respectively for the projections on the first and second coordinates p_1 and p_2, and moreover the following diagram commutes, up to a null set:



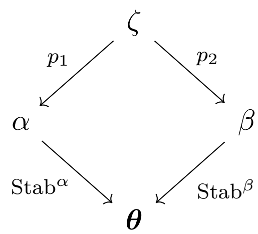
Let θ be an IRS on Γ. We write $\boldsymbol{\theta}$ for the measure-preserving conjugation action

$$\Gamma \overset{\boldsymbol{\theta}}{\curvearrowright} (\mathrm{Sub}(\Gamma), \theta).$$

For every pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$, the map $\mathrm{Stab}^\alpha\colon (X,\mu) \to (\mathrm{Sub}(\Gamma),\theta)$ is an action factor map.

Lemma 2.30 *Let Γ be a countable group and θ be an IRS on Γ. Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$, $\Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$ be pmp actions having IRS θ. Then $\alpha \underset{\boldsymbol{\theta}}{\times} \beta$ has IRS θ.*

Proof Let ζ denote $\alpha \underset{\boldsymbol{\theta}}{\times} \beta$. We know that the following diagram commutes.



Therefore, for $\gamma \in \Gamma$, we have $\forall^*(x, y),\ \gamma x = x \Leftrightarrow \gamma y = y \Leftrightarrow \gamma(x, y) = (x, y)$. It follows that $\forall^*(x, y),\ \mathrm{Stab}^\zeta(x, y) = \mathrm{Stab}^\alpha(x)$ or in other words, $\mathrm{Stab}^\zeta = \mathrm{Stab}^\alpha \circ p_1$. We conclude that

$$\theta_\zeta = \mathrm{Stab}^\zeta_* \eta = \mathrm{Stab}^\alpha_* \left(p_{1*}\eta\right) = \mathrm{Stab}^\alpha_* \mu = \theta_\alpha = \theta. \qquad \square$$

Theorem A states that if α and β are two pmp hyperfinite actions of a group Γ on a standard probability space such that $\theta_\alpha = \theta_\beta$, then α and β are approximately conjugate. We can now prove this theorem:

Proof Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$ be two hyperfinite actions of Γ having IRS θ and consider the joining $\Gamma \overset{\zeta}{\curvearrowright} (Z, \eta)$ from Lemma 2.30.

Applying twice Theorem 2.27 with no Borel parameters we get two pmp bijections $\rho\colon X \to Z$ and $\rho'\colon Y \to Z$ such that:

$$\mu(\{x \in X : \forall i \leq n,\ \rho \circ \gamma_i^\alpha(x) = \gamma_i^\zeta \circ \rho(x)\}) > 1 - \frac{\varepsilon}{2}$$

and

$$\nu(\{y \in Y : \forall i \leq n,\ \rho' \circ \gamma_i^\beta(y) = \gamma_i^\zeta \circ \rho'(y)\}) > 1 - \frac{\varepsilon}{2}.$$

Thus, $\rho'^{-1} \circ \rho\colon X \to Y$ witnesses the ε-approximate conjugacy of α and β. $\square$

3 Model theory of hyperfinite actions

3.1 Measure algebras

The reader unfamiliar with continuous model theory is referred to [3] and Ben Yaacov and Usvyatsov [4]. We will use the same notations as theirs.

Definition 3.1 A *measure algebra* is a Boolean algebra $(\mathcal{A}, \vee, \wedge, \neg, 0, 1, \subseteq, \triangle)$ endowed with a function $\mu\colon \mathcal{A} \to [0,1]$ satisfying the following :

(1) $\mu(1) = 1$.

(2) $\forall a, b \in \mathcal{A},\ \mu(a \wedge b) = 0 \Rightarrow \mu(a \vee b) = \mu(a) + \mu(b)$.

(3) The function $d_\mu(a,b) := \mu(a \triangle b)$ is a complete metric on $\mathcal{A}$.

Proposition 3.2 (Fremlin [11, 323G c)]) *Any measure algebra $\mathcal{A}$ is Dedekind complete, meaning that any subset $S \subseteq \mathcal{A}$ admits a supremum and an infimum, that we respectively denote by $\bigvee S$ and $\bigwedge S$.*

Definition 3.3 An element $a \in \mathcal{A}$ is an *atom* if $\forall b \in \mathcal{A},\ b \subseteq a \Rightarrow b \in \{0, a\}$. A measure algebra is *atomless* if it has no atoms.

We introduce the classical example of a measure algebra: For (X, μ) a probability space, we let $\mathrm{MAlg}(X, \mu)$ be the quotient of the Boolean algebra of measurable subsets of X by the σ-ideal of null sets. For $A \subseteq X$ Borel we denote its class in $\mathrm{MAlg}(X, \mu)$ by $[A]_\mu$. The measure μ descends to the quotient $\mathrm{MAlg}(X, \mu)$ and then $\mathrm{MAlg}(X, \mu)$ endowed with μ is a measure algebra. When (X, μ) is a standard probability space, $\mathrm{MAlg}(X, \mu)$ is atomless and separable for the topology induced by d_μ.

Conversely, we have:

Proposition 3.4 ([11, 331L]) *Let $\mathcal{A}$ be a separable atomless measure algebra. Then there exists a standard probability space (X, μ) such that $\mathcal{A}$ is isomorphic to $\mathrm{MAlg}(X, \mu)$.*

Let $f\colon (X, \mu) \to (Y, \nu)$ be a measure-preserving map. Then the map

$$\widetilde{f}\colon \mathrm{MAlg}(Y, \nu) \to \mathrm{MAlg}(X, \nu)$$

sending $[A]_\nu$ to $\left[f^{-1}(A)\right]_\mu$ is a measure algebra morphism. Moreover, if f is a bimeasurable bijection, then $\widetilde{f}$ is an isomorphism.

However, in general, given a morphism $\varphi\colon \mathrm{MAlg}(X, \nu) \to \mathrm{MAlg}(Y, \mu)$ there is no way to get a lifting of φ, that is a point to point measure-preserving map $\boldsymbol{\varphi}\colon Y \to X$ such that $\widetilde{\boldsymbol{\varphi}} = \varphi$. However, in the case of standard probability spaces, such a construction exists:

Proposition 3.5 (Fremlin [12, 425D]) *Let (X,μ) and (Y,ν) be standard probability spaces. For every morphism of measure algebras $\varphi\colon \mathrm{MAlg}(X,\mu) \to \mathrm{MAlg}(Y,\nu)$ there is a lifting $\boldsymbol{\varphi}\colon Y \to X$ of φ. Moreover, for Γ a countable group acting by automorphisms on $\mathrm{MAlg}(X,\mu)$ by an action α, there is a lifting of α, that is an action $\Gamma \overset{\boldsymbol{\alpha}}{\curvearrowright} X$ by measure-preserving transformations such that $\forall\gamma \in \Gamma$, $\widetilde{\gamma^{\boldsymbol{\alpha}}} = \left(\gamma^{-1}\right)^{\alpha}$.*

Remark 3.6 In the context of the above proposition, such liftings are moreover unique up to a null set, see [11, 343F and 344B].

3.2 Model theory of atomless measure algebras

We axiomatize the theory AMA of atomless measure algebras in the signature $\mathcal{L} = \{\vee, \wedge, \neg, 0, 1\}$ ($\triangle$ is defined as usual) as in [3, Section 16]. The reader unfamiliar with it should also consult a more recent and detailed article by Berenstein and Henson [5].

Proposition 3.7 ([3, Section 16.2]) *The theory* AMA *is separably categorical and therefore complete.*

We also have:

Proposition 3.8 ([3, Sections 16.6 and 16.7]) *The theory* AMA *admits quantifier elimination. Moreover, the definable closure $\mathrm{dcl}^{\mathcal{M}}(C)$ of a subset C in a model $\mathcal{M}$ of* AMA *is the substructure $\langle C \rangle$ of $\mathcal{M}$ generated by C.*

We will now give a characterization of the types in the theory AMA. For that we need a bit of terminology.

To any measure algebra $\mathcal{A}$ we can associate a natural Hilbert space $\mathrm{L}^2(\mathcal{A})$ called the L^2 *space of* $\mathcal{A}$. This construction is consistent in the sense that if $\mathcal{A}$ is the measure algebra of a probability space (X,μ), then there is a natural linear isometry between $\mathrm{L}^2(\mathcal{A})$ and $\mathrm{L}^2(X,\mu)$.

Definition 3.9 Let $\mathcal{A}$ be a measure algebra and $\mathcal{B}$ a measure subalgebra of $\mathcal{A}$. Then the space $\mathrm{L}^2(\mathcal{B})$ is a closed vector subspace of the Hilbert space $\mathrm{L}^2(\mathcal{A})$, we denote by $\mathbb{P}_{\mathcal{B}}$ the orthogonal projection on $\mathrm{L}^2(\mathcal{B})$, and we call it the *conditional expectation* with respect to $\mathcal{B}$. Particularly, for $a \in \mathcal{A}$, a can be seen as the element $\mathbb{1}_a$ of $\mathrm{L}^2(\mathcal{A})$, and we call $\mathbb{P}_{\mathcal{B}}(\mathbb{1}_a)$ the *conditional probability* of a with respect to $\mathcal{B}$. For simplicity, we will denote it by $\mathbb{P}_{\mathcal{B}}(a)$.

By definition, the conditional probability of a with respect to $\mathcal{B}$ is the only $\mathcal{B}$-measurable function such that for any $\mathcal{B}$-measurable function f, we have $\int \mathbb{P}_{\mathcal{B}}(a) f = \int \mathbb{1}_a f$.

Proposition 3.10 ([3, 16.5]) *Let $\mathcal{M} \models$ AMA, $\bar{a}, \bar{b}$ be n-uples of elements of $\mathcal{M}$ and $C \subseteq \mathcal{M}$. Then $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$ if and only if for each $\sigma \colon \{1, \dots, n\} \to \{-1, 1\}$ we have*

$$\mathbb{P}_{\langle C \rangle}\left(\bigwedge_{1 \le i \le n} a_i^{\sigma(i)}\right) = \mathbb{P}_{\langle C \rangle}\left(\bigwedge_{1 \le i \le n} b_i^{\sigma(i)}\right),$$

where a^1 denotes a and a^{-1} denotes its complement $\neg a$ in $\mathcal{M}$.

3.3 The theory $\mathfrak{A}_\theta$

Until now, we studied actions of any countable group. Since any action of a countable group can be represented as an F_∞-action, for the sake of simplicity, we now restrict to F_∞-actions, where F_∞ denotes the countably generated free group.

We now expand the signature $\mathcal{L}$ with a countable set of function symbols indexed by F_∞, that we identify with F_∞ itself, and call this new signature $\mathcal{L}_\infty$. We begin by considering the theory $\mathfrak{A}_{F_\infty}$ consisting of the following axioms:

- The axioms of AMA.
- For $\gamma \in F_\infty$, the axioms expressing that γ is a measure algebra isomorphism:
  - $\sup_{a,b}\ d(\gamma(a \vee b), \gamma a \vee \gamma b) = 0$
  - $\sup_{a,b}\ d(\gamma(a \wedge b), \gamma a \wedge \gamma b) = 0$
  - $\sup_a\ |\mu(\gamma a) - \mu(a)| = 0$
  - $\sup_a\ \inf_b\ d(a, \gamma b) = 0$
- The axioms expressing that F_∞ acts on the measure algebra:
  - $\sup_a\ d(1_{F_\infty} a, a) = 0$
  - For $\gamma_1, \gamma_2 \in F_\infty$, the axiom $\sup_a\ d(\gamma_1(\gamma_2 a), (\gamma_1\gamma_2)a) = 0$

By Propositions 3.4 and 3.5 any separable model of $\mathfrak{A}_{F_\infty}$ can be seen as the action on a measure algebra associated with a measure-preserving action $F_\infty \curvearrowright (X, \mu)$ on a standard probability space. If α is a pmp action on a probability space, we write $\mathcal{M}_\alpha$ for the model of $\mathfrak{A}_{F_\infty}$ induced by α. Without loss of generality, from now on, separable models we consider are always of the form $\mathcal{M}_\alpha$ for α a pmp action on a standard probability space.

Definition 3.11 For f any measure-preserving transformation $(X,\mu)\to(X,\mu)$, where (X,μ) is a probability space, we call the set $\{x\in X : fx\neq x\}$ the *support* of f, and we denote it by $\operatorname{Supp} f$.

Definition 3.12 Let $(\mathcal{A},\mu)$ be a measure algebra, the *support* of an automorphism φ of $\mathcal{A}$ is defined by $\operatorname{supp}\varphi=\bigwedge\{a\in\mathcal{A} : \forall b\subseteq\neg a,\ \varphi b=b\}$.

Let us remark that in our context, $\operatorname{supp}\varphi$ can equivalently be defined as the *minimum* (for the inclusion) of $\{a\in\mathcal{A} : \forall b\subseteq\neg a,\ \varphi b=b\}$ (see [11, Corollary 381F]), a fact that we will not use. The following is also well-known, but we include the proof for completeness.

Lemma 3.13 *Let f be a measure-preserving transformation of a standard probability space (X,μ), then $[\operatorname{Supp} f]_\mu=\operatorname{supp}\widetilde{f}$.*

Proof Since f fixes pointwise the elements $x\notin\operatorname{Supp} f$, in particular it fixes setwise the subsets of the complement of $\operatorname{Supp} f$. So by definition of the measure algebra support, we have $\operatorname{supp}\widetilde{f}\subseteq[\operatorname{Supp} f]_\mu$.

Conversely, fix a countable family of Borel subsets (C_n) which separate the points of X, meaning that if $x\neq y\in X$ there is n such that $x\in C_n$ but $y\notin C_n$. Then if $f(x)\neq x$, we find n such that $x\in C_n$ but $f(x)\notin C_n$, which means that $x\in C_n\setminus f^{-1}(C_n)$. In particular, elements of the form $D_n=C_n\setminus f^{-1}(C_n)$ cover $\operatorname{Supp} f$ and satisfy $D_n\cap f(D_n)=\emptyset$, which means that $[\operatorname{Supp} f]$ is covered by elements d_n such that $d_n\wedge\widetilde{f}(d_n)=0$.

Now take $a\in\mathcal{A}$ such that $\forall b\subseteq\neg a, \widetilde{f}(b)=b$. Assume by contradiction $\neg a$ does not contain $[\operatorname{Supp} f]$, then $\neg a$ must intersect some d_n, but then $\widetilde{f}(d_n\wedge\neg a)$ is disjoint from d_n, in particular it is different from the nonzero element $d_n\wedge\neg a$, a contradiction to the definition of a. So we must have $[\operatorname{Supp} f]\subseteq a$, and since $\operatorname{supp}\widetilde{f}=\bigwedge\{a\in\mathcal{A} : \forall b\subseteq\neg a,\ \widetilde{f}(b)=b\}$, we conclude that the reverse inclusion $[\operatorname{Supp} f]\subseteq\operatorname{supp}\varphi$ holds as wanted. □

Our goal is now to give a first order description of the support of an automorphism of a separable measure algebra:

Lemma 3.14

(1) *Let φ be an automorphism of a measure algebra $\mathcal{A}$ such that $\operatorname{supp}\varphi\neq 0$. Then there exists $b\neq 0\in\mathcal{A}$ such that $\varphi b\wedge b=0$.*

(2) *Let $\mathcal{A}$ be a measure algebra. Let φ be an automorphism of $\mathcal{A}$.*
Then there is $a_0 \in \mathcal{A}$ such that $\operatorname{supp} \varphi = \varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$ and $a_0 \wedge \varphi a_0 = 0$.
Furthermore, we have $\operatorname{supp} \varphi = \bigvee\{\varphi^{-1}a \vee a \vee \varphi a : a \in \mathcal{A}, a \wedge \varphi a = 0\}$.

Proof (1): By construction, there must exist $c \subseteq \operatorname{supp} \varphi$ such that $\varphi c \neq c$. Let $b = c \setminus \varphi^{-1}(c) \neq 0$. Then $\varphi(b) = \varphi(c) \setminus c$ is disjoint from c by construction, in particular it is disjoint from b as desired.

(2): First $\mathcal{A}$ is a measure algebra and therefore is complete as a Boolean algebra so by Zorn's lemma it has a maximal element a_0 disjoint from its image by φ.

Consider $b = \varphi^2 a_0 \setminus (\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0)$. We have

$$\begin{aligned} (a_0 \vee b) \wedge \varphi(a_0 \vee b) &= (a_0 \wedge \varphi a_0) \vee (a_0 \wedge \varphi b) \vee (b \wedge \varphi a_0) \vee (b \wedge \varphi b) \\ &\subseteq 0 \vee (a_0 \setminus a_0) \vee (\varphi a_0 \setminus \varphi a_0) \vee (\varphi^2 a_0 \setminus \varphi^2 a_0) \\ &= 0. \end{aligned}$$

Thus, $a_0 \vee b$ is disjoint from its image. By maximality of a_0, we then have $b \subseteq a_0$, but by definition $b \wedge a_0 = 0$, so $b = 0$, or in other words, $\varphi^2 a_0 \subseteq \varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$.

It follows that $\varphi\left(\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0\right) \subseteq \varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$ and since φ preserves the measure, the set $\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$ is invariant by φ.

Furthermore, a_0 is disjoint from its image by φ, and so $\varphi^{-1}a_0$ and φa_0 are also disjoint from their respective image, so we have

$$\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0 \subseteq \operatorname{supp} \varphi.$$

Conversely, let $c = \operatorname{supp} \varphi \setminus (\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0)$ and suppose that $c \neq 0$. Since c is invariant by φ, we can consider the automorphism $\varphi \restriction_c$ of the measure algebra lying under c. Applying the first point of this lemma to this automorphism, we get a nontrivial $b \subseteq c$ disjoint from its image by φ.

But then, $a_0 \vee b$ contradicts the maximality of a_0. We conclude that

$$\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0 = \operatorname{supp} \varphi.$$

Finally, as we already noticed, any set of the form $\varphi^{-1}a \vee a \vee \varphi a$ for $a \wedge \varphi a = 0$ is a subset of $\operatorname{supp} \varphi$, so we have

$$\operatorname{supp} \varphi = \bigvee\{\varphi^{-1}a \vee a \vee \varphi a : a \in \mathcal{A}, a \wedge \varphi a = 0\}. \qquad \square$$

Now we can prove that the IRS of a pmp action on a measure algebra is determined by the theory of this action seen as a model of $\mathfrak{A}_{F_\infty}$.

Definition 3.15 For $\gamma \in F_\infty$ we let $t_\gamma(a)$ denote the term $\gamma^{-1}(a \setminus \gamma a) \vee (a \setminus \gamma a) \vee \gamma(a \setminus \gamma a)$.

It follows from Lemma 3.14 that for $\mathcal{M} \models \mathfrak{A}_{F_\infty}$, $\operatorname{supp} \gamma = \bigvee\{t_\gamma(a) : a \in \mathcal{M}\}$, and there is a_γ such that

$$\operatorname{supp} \gamma = t_\gamma(a_\gamma).$$

We can now present the theory which axiomatizes pmp actions having IRS θ, motivated by Lemma 2.22 and the previous observation: For θ an IRS, let $\mathfrak{A}_\theta$ be the $\mathcal{L}_\infty$-theory consisting of:

- The axioms of $\mathfrak{A}_{F_\infty}$;
- For $F \subseteq F_\infty$ finite, the axiom $\sup_{\{a_\gamma : \gamma \in F\}} \mu(\bigwedge_{\gamma \in F} t_\gamma(a_\gamma)) = \theta(\mathrm{N}_{F,\varnothing})$,

where we recall that $\mathrm{N}_{F,G} = \{\Lambda \leq F_\infty : F \subseteq \Lambda, G \cap \Lambda = \varnothing\}$.

Theorem 3.16 *The separable models of $\mathfrak{A}_\theta$ are exactly the measure-preserving actions of F_∞ on standard probability spaces which have IRS θ.*

Proof Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ be a pmp action satisfying the theory $\mathfrak{A}_\theta$ when viewed as an action α by automorphisms on the separable atomless measure algebra $\mathrm{MAlg}(X, \mu)$, and let θ' be its IRS. For a finite subset $F \subseteq \Gamma$, we have by the definition of θ' and Lemma 3.13 that :

$$\theta'(\mathrm{N}_{F,\emptyset}) = \mu(\bigcap_{\gamma \in F} \operatorname{Supp}(\gamma^{\boldsymbol{\alpha}})) = \mu(\bigwedge_{\gamma \in F} \operatorname{supp}(\gamma^\alpha)),$$

and by the second item of Lemma 3.14 we have

$$\mu(\bigwedge_{\gamma \in F} \operatorname{supp}(\gamma^\alpha)) = \sup_{\{a_\gamma : \gamma \in F\}} \mu(\bigwedge_{\gamma \in F} t_\gamma(a_\gamma)) = \theta(\mathrm{N}_{F,\varnothing}),$$

which finishes the proof by a direct application of Lemma 2.22. □

The following noteworthy corollary is then immediate.

Corollary 3.17 *Let $\mathcal{M}_\alpha, \mathcal{M}_\beta$ be two elementarily equivalent separable models of $\mathfrak{A}_{F_\infty}$. Then $\theta_\alpha = \theta_\beta$.* □

Let us finally note that the term t_γ given by Definition 3.15 can also be used to show the definability of supports in the following sense.

Lemma 3.18 *Let $\gamma \in F_\infty$. Then the support of γ is definable without parameters in the theory $\mathfrak{A}_{F_\infty}$. Precisely, the distance to $\mathrm{supp}\ \gamma$ is defined by $d(a, \mathrm{supp}\ \gamma) = \inf_b\ \mu(a \setminus t_\gamma(b)) + \sup_b\ \mu(t_\gamma(b) \setminus a)$.*

Proof By definition of the distance, we have $\forall a \in \mathcal{M},\ d(a, \mathrm{supp}\ \gamma) = \mu(a \setminus \mathrm{supp}\ \gamma) + \mu(\mathrm{supp}\ \gamma \setminus a)$.

Using Lemma 3.14, we have on the one hand $\mu(a \setminus \mathrm{supp}\ \gamma) = \inf_b\ \mu(a \setminus t_\gamma(b))$ and on the other hand, $\mu(\mathrm{supp}\ \gamma \setminus a) = \sup_b\ \mu(t_\gamma(b) \setminus a)$, which completes the proof. □

3.4 Completeness and Model Completeness

Definition 3.19 Let (X, μ) be a standard probability space and Γ be a countable group.

First, let $\mathrm{Aut}(X, \mu)$ be the space of automorphisms of $\mathrm{MAlg}(X, \mu)$. We equip it with a complete metric d_u called the *uniform metric* and defined by the formula $d_u(\varphi, \psi) := \sup_{a \in \mathrm{MAlg}(X,\mu)} d_\mu(\varphi a, \psi a)$. We call the topology induced the *uniform topology*.

Then we define the space $A(\Gamma, X, \mu)$ of pmp actions of Γ on (X, μ) naturally as a subspace of $\mathrm{Aut}(X, \mu)^\Gamma$. The uniform topology on $\mathrm{Aut}(X, \mu)$ gives rise to a product topology on $\mathrm{Aut}(X, \mu)^\Gamma$ which is completely metrizable and for which $A(\Gamma, X, \mu)$ is closed. Again, we call this topology the *uniform topology* on $A(\Gamma, X, \mu)$.

From now on, fix a complete metric d_u compatible with the uniform topology on $A(F_\infty, X, \mu)$.

Theorem 3.20 *Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}_\infty$-formula, where $|\bar{x}| = n$, $|\bar{y}| = m$, let (X, μ) be a standard probability space and let $\bar{p} \in \mathrm{MAlg}(X, \mu)^m$.*

Then the map

$$\begin{array}{ccc} (A(F_\infty, X, \mu), d_u) & \longrightarrow & \left(\ell^\infty(\mathrm{MAlg}(X, \mu)^n), \|\ \|_\infty\right) \\ \alpha & \longmapsto & \left(\varphi^{\mathcal{M}_\alpha}(\bar{a}, \bar{p})\right)_{\bar{a} \in \mathrm{MAlg}(X,\mu)^n} \end{array}$$

is uniformly continuous.

Proof We prove this result by induction on formulas. For now assume that the theorem holds for atomic formulas. First remark that if the theorem holds for certain formulas, then it holds for any combination of these formulas constructed with the help of connectives, by using their uniform continuity. Then it suffices to treat the case of quantifiers to conclude. But it is immediate, since we use the norm $\| \|_\infty$.

Let us now prove the theorem for atomic formulas. If $\varphi(\bar{x}, \bar{y})$ is an atomic formula, then it is equivalent to a formula of the form $\varphi(\bar{x}, \bar{y}) \coloneqq \mu(t(\gamma_1\bar{x}, \dots, \gamma_l\bar{x}, \gamma_1\bar{y}, \dots, \gamma_l\bar{y}))$ for an $\mathcal{L}$-term t and some $\gamma_1, \dots, \gamma_l \in F_\infty$. Let $\varepsilon > 0$.

By definition of the terms, they are uniformly continuous and so there is $\delta > 0$ such that for $\bar{z}$ and $\bar{z'} \in \mathrm{MAlg}(X, \mu)^{(n+m)l}$, if $d_\mu(\bar{z}, \bar{z'}) < \delta$ then $d_\mu(t(\bar{z}), t(\bar{z'})) < \varepsilon$.

Now if $\alpha, \beta \in A(F_\infty, X, \mu)$ are sufficiently d_u-close, then for every $a \in \mathrm{MAlg}(X, \mu)$ and $1 \leq i \leq l$, $d_\mu(\gamma_i^\alpha a, \gamma_i^\beta a) < \delta$. It follows that for all $\bar{a} \in \mathrm{MAlg}(X, \mu)^n$,

$$\begin{aligned} \left| \varphi^{\mathcal{M}_\alpha}(\bar{a}, \bar{p}) \right. & \left. - \varphi^{\mathcal{M}_\beta}(\bar{a}, \bar{p}) \right| \\ & \leq d_\mu \left(t(\gamma_1^\alpha \bar{a}, \dots, \gamma_l^\alpha \bar{a}, \gamma_1^\alpha \bar{p}, \dots, \gamma_l^\alpha \bar{p}), t(\gamma_1^\beta \bar{a}, \dots, \gamma_l^\beta \bar{a}, \gamma_1^\beta \bar{p}, \dots, \gamma_l^\beta \bar{p}) \right) \\ & < \varepsilon, \end{aligned}$$

which finishes the proof. □

Theorem 3.21 *Let θ be a hyperfinite IRS on F_∞. Then the theory $\mathfrak{A}_\theta$ is model complete.*

Proof It suffices to show that any inclusion of two separable models is elementary. Indeed, suppose this result and take any $\mathcal{M} \subseteq \mathcal{N} \models \mathfrak{A}_\theta$, $\varphi(\bar{x})$ a $\mathcal{L}_\infty$-formula and $\bar{p} \in \mathcal{M}$ finite. By the Löwenheim–Skolem theorem, find a separable $\mathcal{M}' \preceq \mathcal{M}$ containing $\bar{p}$. Again by the Löwenheim–Skolem theorem, find a separable $\mathcal{N}' \preceq \mathcal{N}$ containing the separable structure $\mathcal{M}'$. Using the hypothesis, $\mathcal{M}' \preceq \mathcal{N}'$, so we finally get

$$\varphi(\bar{p})^{\mathcal{M}} = \varphi(\bar{p})^{\mathcal{M}'} = \varphi(\bar{p})^{\mathcal{N}'} = \varphi(\bar{p})^{\mathcal{N}}.$$

Let $\mathcal{M} \subseteq \mathcal{N}$ be two separable models of $\mathfrak{A}_\theta$. Consider a $\mathcal{L}_\infty$-formula $\varphi(\bar{x})$ with k variables and $\bar{p} \in \mathrm{MAlg}(X, \mu)^k$.

A classical argument derived from Proposition 3.5 allows us to choose two pmp actions $F_\infty \overset{\alpha}{\curvearrowright} (X, \mu)$ and $F_\infty \overset{\beta}{\curvearrowright} (Y, \nu)$ on standard probability spaces along with a pmp map $\pi : Y \to X$, such that $\mathcal{M} \simeq \mathcal{M}_\alpha$, $\mathcal{N} \simeq \mathcal{M}_\beta$, and π is a lifting of the inclusion $\mathrm{MAlg}(X, \mu) \hookrightarrow \mathrm{MAlg}(Y, \nu)$, which is equivariant respectively to the actions α and β. For $1 \leq i \leq k$, let $A_i \subseteq X$ be a Borel representative of p_i and let $B_i = \pi^{-1}(A_i)$, which is also a Borel representative of p_i, in Y.

Then by Theorem 2.27, α is in the uniform closure of the set

$$\mathcal{C}(\beta) := \{\rho^{-1}\beta\rho : \rho \text{ is a pmp bijection } X \to Y \text{ such that } \forall i \leq k, \rho^{-1}(A_i) = B_i\}.$$

But then Theorem 3.20 implies that $\varphi^{\mathcal{M}_\alpha}(\bar{p}) \in \overline{\{\varphi^{\mathcal{M}_{\beta'}}(\bar{p}) : \beta' \in \mathcal{C}(\beta)\}}$. Furthermore, for any $\beta' \in \mathcal{C}(\beta)$, we have $(\beta', \bar{A}) \simeq (\beta, \bar{B})$, so that $(\mathcal{M}_{\beta'}, \bar{p}) \equiv (\mathcal{M}_\beta, \bar{p})$ and consequently $\varphi^{\mathcal{M}_{\beta'}}(\bar{p}) = \varphi^{\mathcal{M}_\beta}(\bar{p})$. This establishes that $\varphi^{\mathcal{M}_\alpha}(\bar{p}) = \varphi^{\mathcal{M}_\beta}(\bar{p})$.

Hence $\mathcal{M}_\alpha \preceq \mathcal{M}_\beta$, and therefore $\mathfrak{A}_\theta$ is model complete. □

Now for completeness we combine model completeness with the argument of amalgamation already seen in Subsection 2.5.2.

Theorem 3.22 *Let θ be a hyperfinite IRS on F_∞. Then the theory $\mathfrak{A}_\theta$ is complete.*

Proof As usual, it is sufficient to prove that two separable models of $\mathfrak{A}_\theta$ are elementarily equivalent.

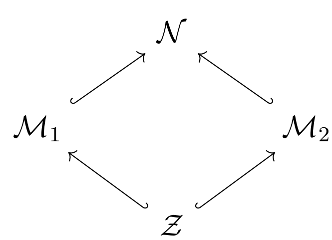
Let $\mathcal{M}_\alpha, \mathcal{M}_\beta \models \mathfrak{A}_\theta$ be two separable models and consider the action $\zeta := \alpha \underset{\theta}{\times} \beta$. By Lemma 2.30, we have $\mathcal{M}_\zeta \models \mathfrak{A}_\theta$ and moreover, both $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are substructures of $\mathcal{M}_\zeta$.

Now since $\mathfrak{A}_\theta$ is model complete, we have $\mathcal{M}_\alpha \preceq \mathcal{M}_\zeta$ and $\mathcal{M}_\beta \preceq \mathcal{M}_\zeta$, so $\mathcal{M}_\alpha \equiv \mathcal{M}_\zeta \equiv \mathcal{M}_\beta$. □

3.5 Elimination of quantifiers

Proposition 3.23 ([3, Proposition 13.16]) *Let T be a countable theory. Then T admits quantifier elimination if and only if for any $\mathcal{M}, \mathcal{N} \models T$, any substructure $\mathcal{Z} \subseteq \mathcal{M}$ and any embedding $f\colon \mathcal{Z} \hookrightarrow \mathcal{N}$, there is an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ and an embedding $\tilde{f}\colon \mathcal{M} \hookrightarrow \mathcal{N}'$ extending f.*

Definition 3.24 We say that a theory T admits *amalgamation* if for any $\mathcal{M}_1, \mathcal{M}_2 \models T$ and any common substructure $\mathcal{Z}$, there is a joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$, that is a structure $\mathcal{N} \models T$ and embeddings $\mathcal{M}_i \hookrightarrow \mathcal{N}$ $(i = 1, 2)$ such that the following diagram commutes:



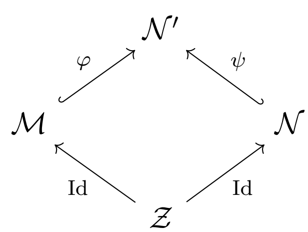
The next lemma is a classical result in discrete model theory, and it easily extends to continuous model theory.

Lemma 3.25 *Let T be a theory. Then T admits quantifier elimination if and only if it admits amalgamation and is model complete.*

Proof Suppose that T admits quantifier elimination. Let $\mathcal{M}_1, \mathcal{M}_2 \models T$ with a common substructure $\mathcal{Z}$, applying Proposition 3.23 where f is the inclusion $\mathcal{Z} \hookrightarrow \mathcal{M}_2$, we get $\mathcal{N}$ as required.

Now let $\mathcal{M} \subseteq \mathcal{N}$ be two models of T. By quantifier elimination, we only need to prove that $\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(\bar{a})$ for atomic formulas φ and finite tuples $\bar{a}$ of parameters in $\mathcal{M}$. But this is trivial by the definition of inclusion for models.

Conversely, suppose T admits amalgamation and is model complete and let $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{Z} \subseteq \mathcal{M}$ be a substructure, and $f \colon \mathcal{Z} \hookrightarrow \mathcal{N}$. By considering a monster model, we may suppose that $\mathcal{Z} \subseteq \mathcal{N}$ is a substructure and f is the identity. Then by amalgamation there is a model $\mathcal{N}' \models T$ and embeddings φ, ψ such that the following diagram commutes:



Again we may suppose that $\mathcal{N} \subseteq \mathcal{N}'$ and ψ is the identity, thus by model completeness we have $\mathcal{N} \preceq \mathcal{N}'$. Furthermore, the diagram now exactly states that φ extends the inclusion $\mathcal{Z} \hookrightarrow \mathcal{N}$. □

In order to prove that our theories eliminate quantifiers, it only remains to prove that they have amalgamation. However, the following example shows that this is not the case in general.

Definition 3.26 Let $\Gamma \overset{\alpha}{\curvearrowright} X$ be an action of a group on a standard Borel space. We say that a Γ-invariant measure $\mu \in \mathfrak{P}(X)$ is *ergodic* if every measurable subset of X which is α-invariant must be either null or conull for μ.

Observe that any non-ergodic Γ-invariant probability measure μ is a nontrivial convex combination of two other such measures: if A is α-invariant and neither null nor conull then $\mu = \mu(A)\mu_A + (1-\mu(A))\mu_{X\setminus A}$, where $\mu_B(C) = \frac{1}{\mu(B)}\mu(B \cap C)$.

Remark 3.27 It can be shown that conversely, when the standard space structure of X comes from a compact metrizable topology and the Γ-action α is by homeomorphisms, ergodic measures are exactly the extreme points of the compact convex space $\mathfrak{P}^{\alpha}(X)$ of α-invariant probability measures. Moreover, the convex structure on $\mathfrak{P}^{\alpha}(X)$ does not depend on this topology, and every pmp action on a standard probability space can be realized as an action by homeomorphisms on a compact metrizable space (see [13, Theorem 2.15]), so ergodic measures always correspond to extreme points of $\mathfrak{P}^{\alpha}(X)$.

For Invariant Random Subgroups, we consider the notion of ergodicity always with respect to the action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by conjugation.

Proposition 3.28 *Let θ be a non-ergodic IRS on F_∞. Then $\mathfrak{A}_\theta$ does not have quantifier elimination.*

Proof Take any finite subset $F \subseteq F_\infty$. Then

$$\mu\left(x \wedge \bigwedge_{\gamma \in F} \mathrm{supp}\,\gamma\right) := \sup_{\{a_\gamma : \gamma \in F\}} \mu\left(x \wedge \bigwedge_{\gamma \in F} t_\gamma(a_\gamma)\right)$$

is a definable predicate in the signature $\mathcal{L}_\infty$. However, as we shall see, not all predicates of this form are definable without quantifiers.

Indeed, suppose that for every finite subset $F \subseteq F_\infty$, there is a quantifier free formula $\varphi_F(x)$ equivalent to $\mu\left(x \wedge \bigwedge_{\gamma \in F} \mathrm{supp}\,\gamma\right)$.

Since θ is not ergodic, as noted right after Definition 3.26 we have $\theta = t\theta_1 + (1-t)\theta_2$ for a $t \in (0, \frac{1}{2}]$ and $\theta_1 \neq \theta_2$ two IRSs on F_∞. Let κ_1 be a pmp action on $([0,1], \lambda)$ having IRS θ_1 and κ_2 be a pmp action on $([0,1], \lambda)$ having IRS θ_2. Define

- $F_\infty \overset{\alpha}{\curvearrowright} (X = [0,1] \times \{1,2,3\}, \mu = t\lambda \times \delta_1 + t\lambda \times \delta_2 + (1-2t)\lambda \times \delta_3)$ that acts like κ_1 on $[0,1] \times \{1\}$ and acts like κ_2 both on $[0,1] \times \{2\}$ and on $[0,1] \times \{3\}$.
- $F_\infty \overset{\beta}{\curvearrowright} (X = [0,1] \times \{1,2,3\}, \mu = t\lambda \times \delta_1 + t\lambda \times \delta_2 + (1-2t)\lambda \times \delta_3)$ that acts like κ_1 on $[0,1] \times \{2\}$ and acts like κ_2 both on $[0,1] \times \{1\}$ and on $[0,1] \times \{3\}$.

We have $\theta_\alpha = \theta_\beta = \theta$.

Let $\mathcal{M}$ be the finite measure algebra generated by three atoms $\{a, b, c\}$ of respective measure t, t and $1-2t$. By sending a to $[0,1]\times\{1\}$, b to $[0,1]\times\{2\}$ and c to $[0,1]\times\{3\}$, one can embed $\mathcal{M}$ in both $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$. Then $\mathcal{M}$ endowed with the trivial action is a common substructure of $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$.

As $\varphi_F(x)$ is quantifier free, we have $\varphi_F^{\mathcal{M}_\alpha}(a) = \varphi_F^{\mathcal{M}}(a) = \varphi_F^{\mathcal{M}_\beta}(a)$, but

$$\mathcal{M}_\alpha \models \mu(a \wedge \bigwedge_{\gamma\in F} \operatorname{supp}\gamma) = t\theta_1(\mathrm{N}_F) \text{ whereas } \mathcal{M}_\beta \models \mu(a \wedge \bigwedge_{\gamma\in F} \operatorname{supp}\gamma) = t\theta_2(\mathrm{N}_F).$$

Since an IRS is determined by its values on the sets of the form N_F, we get $\theta_1 = \theta_2$, a contradiction. □

Thus, non-ergodicity of the IRS is an obstacle to quantifier elimination. A natural question is to ask about a converse:

> *For which θ does the theory $\mathfrak{A}_\theta$ admit quantifier elimination? Is it the case for any ergodic IRS?*

The author does not have any satisfying answer. However, we answer another interesting question. One can ask what we can reasonably add to the theory $\mathfrak{A}_\theta$ to expand it into a theory $\mathfrak{A}'_\theta$ in a signature $\mathcal{L}'_\infty \supseteq \mathcal{L}_\infty$ which has quantifier elimination.

The issue encountered in Proposition 3.28 is that formulas involving the supports of the elements of F_∞ may not be equivalent to quantifiers free formulas in $\mathfrak{A}_\theta$. This motivates us to look at expansions that allow us to talk about the supports of elements of F_∞ in the language. For that we add constants $\{S_\gamma : \gamma \in F_\infty\}$ to the signature $\mathcal{L}_\infty$ to get a new signature $\mathcal{L}'_\infty$, and we consider the theory $\mathfrak{A}'_\theta$ consisting of:

- The axioms of $\mathfrak{A}_\theta$.
- For $\gamma \in F_\infty$, the axioms:
  - { $\sup_a d(S_\gamma \wedge t_\gamma(a), t_\gamma(a)) = 0$.
  - { $\mu(S_\gamma) = \theta(\mathrm{N}_\gamma)$.

This theory expresses that for $\gamma \in F_\infty$, the constant S_γ must be interpreted as $\operatorname{supp}\gamma^{\mathcal{M}}$ in the model $\mathcal{M}$, as it contains the support by the first axiom and has the same measure by the second one.

We need a last definition in order to prove that the theories $\mathfrak{A}_\theta$ admit amalgamation for θ hyperfinite:

Definition 3.29 Let $\mathcal{M} \models \mathfrak{A}_\theta$, we denote by $\mathcal{I}_\mathcal{M}$ and we call the IRS of $\mathcal{M}$ the substructure of $\mathcal{M}$ generated by the elements $\operatorname{supp}\gamma$ for $\gamma \in \Gamma$.

Note that this naming is consistent: let $\mathcal{M} = \mathcal{M}_\alpha$ for a pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ having IRS θ. Then $\mathcal{I}_\mathcal{M}$ is isomorphic to the measure algebra $\mathcal{I}_\theta$ associated to the action $\Gamma \overset{\boldsymbol{\theta}}{\curvearrowright} (\mathrm{Sub}(\Gamma), \theta)$ and moreover, the map $\mathrm{Stab}^\alpha : X \to \mathrm{Sub}(\Gamma)$ is a lifting of the inclusion $\mathcal{I}_\mathcal{M} \subseteq \mathcal{M}$.

Theorem 3.30 *Let θ be an IRS, then the theory $\mathfrak{A}'_\theta$ admits amalgamation in the signature $\mathcal{L}'_\infty$.*

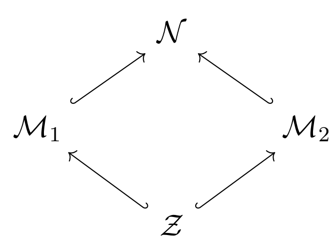
Proof Let $\mathcal{M}_1, \mathcal{M}_2 \models \mathfrak{A}'_\theta$ and let $\mathcal{Z}$ be a common substructure of $\mathcal{M}_1$ and $\mathcal{M}_2$. Then by definition of the theory $\mathfrak{A}'_\theta$, $\mathcal{I}_\theta$ is a substructure of $\mathcal{Z}$ and the inclusions $\mathcal{Z} \hookrightarrow \mathcal{M}_1$ and $\mathcal{Z} \hookrightarrow \mathcal{M}_2$ send $\mathcal{I}_\theta$ on $\mathcal{I}_{\mathcal{M}_1}$ and $\mathcal{I}_{\mathcal{M}_2}$ respectively. For the sake of simplicity, we identify $\mathcal{Z}$ with its images in $\mathcal{M}_1$ and $\mathcal{M}_2$, which implies that $\mathcal{I}_\theta$, $\mathcal{I}_{\mathcal{M}_1}$ and $\mathcal{I}_{\mathcal{M}_2}$ are all identified.

Let X_1, X_2 and Z be the respective Stone spaces of $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{Z}$ (see [11, 321J]) and let μ_1, μ_2 be the respective inner regular Borel probability measures on X_1 and X_2 so that F_∞ naturally acts on each (X_i, μ_i) in a measure-preserving fashion (see [11, 324E]). We define an inner regular Borel probability measure ν on $X_1 \times X_2$ as in Ben Yaacov [2, Construction 2.3] as the continuous extension of the map defined on cylinders by the formula:

$$\nu(a_1 \times a_2) = \int_Z \mu_1(a_1|\mathcal{Z})\mu_2(a_2|\mathcal{Z})\, dz \text{ for all } a_1 \in \mathcal{M}_1, a_2 \in \mathcal{M}_2.$$

The pmp action $F_\infty \curvearrowright (X_1 \times X_2, \nu)$ then induces a structure $\mathcal{N} \models \mathfrak{A}_{F_\infty}$ that we call the *relative independent joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$*.

The following diagram is indeed commutative:



It remains to prove that $\mathcal{N} \models \mathfrak{A}_\theta$. For that note that

$$\begin{aligned} \neg \operatorname{supp} \gamma^{\mathcal{N}} &= \bigvee \{a : \forall b \subseteq a, \gamma b = b\} \\ &= \bigvee \{a_1 \times a_2 : \forall b \subseteq a_1 \times a_2, \gamma b = b\} \\ &= \bigvee \{a_1 \times a_2 : \forall b_1 \subseteq a_1 \ \forall b_2 \subseteq a_2, \gamma b_1 = b_1 \text{ and } \gamma b_2 = b_2\}, \end{aligned}$$

where the second and third equalities rely on the fact that every element of the measure algebra of (Z, ν) is the supremum of elements of the form $a_1 \times a_2$, see [12, 325D(c)(ii)]. Hence we have

$$\begin{aligned} \neg \operatorname{supp} \gamma^{\mathcal{N}} &= \neg \operatorname{supp} \gamma^{\mathcal{M}_1} \times \neg \operatorname{supp} \gamma^{\mathcal{M}_2} \\ &= \neg S_\gamma^{\mathcal{Z}} \times \neg S_\gamma^{\mathcal{Z}} \end{aligned}$$

but the definition of ν implies that $\nu\left(\neg S_\gamma^{\mathcal{Z}} \times 1_{\mathcal{M}_2}\right) = \nu\left(\neg S_\gamma^{\mathcal{Z}} \times \neg S_\gamma^{\mathcal{Z}}\right)$, so that these two elements of $\mathcal{N}$ are equal. Letting i_1 denote the embedding $\mathcal{M}_1 \hookrightarrow \mathcal{N}$, we get the equalities $\neg \operatorname{supp} \gamma^{\mathcal{N}} = \neg S_\gamma^{\mathcal{Z}}$ and therefore $\operatorname{supp} \gamma^{\mathcal{N}} = i_1\left(S_\gamma^{\mathcal{Z}}\right) = i_1\left(\operatorname{supp} \gamma^{\mathcal{M}_1}\right)$. This being true for any $\gamma \in F_\infty$, it follows that i_1 maps any finite intersection of supports in $\mathcal{M}_1$ to the corresponding intersection of supports in $\mathcal{N}$, and since i_1 also preserves the measure, we can conclude that $\mathcal{N} \models \mathfrak{A}_\theta$. □

Theorem 3.31 *Let θ be a hyperfinite IRS. Then the theory $\mathfrak{A}'_\theta$ eliminates quantifiers in the signature $\mathcal{L}'_\infty$.*

Proof We use Lemma 3.25.

We just saw that $\mathfrak{A}'_\theta$ admits amalgamation.

For model completeness, take $\mathcal{M} \subseteq \mathcal{N}$ be two models of $\mathfrak{A}'_\theta$ and let us prove that $\mathcal{M} \preceq \mathcal{N}$. Let $\varphi(\bar{x})$ be an $\mathcal{L}'_\infty$-formula and $\bar{p} \in \mathcal{M}^n$. Then $\varphi(\bar{x})$ is equivalent to a formula of the form $\psi(\bar{x}, S_{\bar{\gamma}})$ where ψ is a $\mathcal{L}_\infty$-formula, and the constants of the form S_γ are preserved under the inclusion $\mathcal{M} \subseteq \mathcal{N}$. Therefore, it suffices to apply Theorem 3.21 to ψ and to consider the elements $S_{\bar{\gamma}}$ as parameters added to $\bar{p}$ to conclude. □

As a corollary, we get a class of IRSs θ for which the theory $\mathfrak{A}_\theta$ admits quantifier elimination.

Corollary 3.32 *The theory of free actions of an amenable group admits amalgamation. Namely, if θ is the Dirac measure δ_N for a co-amenable normal subgroup $N \leq F_\infty$, then $\mathfrak{A}_\theta$ has quantifier elimination.*

Proof Simply note that the support of an element $\gamma \in F_\infty$ in a model of $\mathfrak{A}_\theta$ is either 0 (if $\gamma \in N$) or 1 (if $\gamma \notin N$). It follows that the theories $\mathfrak{A}_\theta$ and $\mathfrak{A}'_\theta$ completely coincide, hence the result. □

For $\mathcal{M} \models \mathfrak{A}_\infty$ and $A \subseteq \mathcal{M}$, we write $\langle A \rangle$ for the closed subalgebra of $\mathcal{M}$ (that is, the substructure of $\mathcal{M}$ as a model of AMA) generated by A.

Theorem 3.33 *Let θ be a hyperfinite IRS. Let $\mathcal{M} \models \mathfrak{A}_\theta$ and $A \subseteq \mathcal{M}$. Then the definable closure of A in $\mathcal{M}$ is $\langle F_\infty A \cup \mathcal{I}_\mathcal{M} \rangle$.*

Proof On the one hand, $A \subseteq \mathrm{dcl}^{\mathcal{M}}(A)$ and by Lemma 3.18, for $\gamma \in F_\infty$, $\mathrm{supp}\, \gamma^{\mathcal{M}} \in \mathrm{dcl}^{\mathcal{M}}(A)$. Thus we get the first inclusion.

On the other hand, since $\mathfrak{A}'_\theta$ expands $\mathfrak{A}_\theta$, the definable closure of A in the theory $\mathfrak{A}_\theta$ is contained in the definable closure of A in the theory $\mathfrak{A}'_\theta$. Let us compute this definable closure D.

First, we notice that the function symbols γ are interpreted by automorphisms and thus any atomic $\mathcal{L}_\infty$-formula with parameters in A is equivalent to an atomic $\mathcal{L}$-formula with parameters in $F_\infty A$. This remark then extends to quantifier free formulas.

Then, by Theorem 3.31, any $\mathcal{L}'_\infty$-formula with parameters in A is equivalent to a quantifier free $\mathcal{L}'_\infty$-formula with parameters in A and since we only added constants in $\mathcal{L}_\infty$, it is moreover equivalent to a quantifier free $\mathcal{L}_\infty$-formula with parameters in $A \cup \mathcal{I}_\mathcal{M}$.

Combining the two latter properties and the fact that $\mathrm{dcl}(A) = \langle A \rangle$ in the theory AMA, we get that $D = \langle F_\infty(A \cup \mathcal{I}_\mathcal{M}) \rangle$. Furthermore, $\mathcal{I}_\mathcal{M}$ is a substructure and so $\langle F_\infty(A \cup \mathcal{I}_\mathcal{M}) \rangle = \langle F_\infty A \cup \mathcal{I}_\mathcal{M} \rangle$.

Hence the conclusion. □

3.6 Stability and Independence

We recall some definitions from [3].

Definition 3.34 Let κ be a cardinal. A *κ-universal domain* for a theory T is a κ-saturated and strongly κ-homogeneous model of T. If $\mathcal{U}$ is a κ-universal domain and $A \subseteq \mathcal{U}$, we say that A is *small* if $|A| < \kappa$.

Definition 3.35 Let $\mathcal{U}$ be a κ-universal domain for T. A *stable independence relation* on $\mathcal{U}$ is a relation $A \mathop{\smile\!\!\!\!\!\!|}_C B$ on triples of small subsets of $\mathcal{U}$ satisfying the following properties, for all small $A, B, C, D \subseteq \mathcal{U}$, finite $\bar{u}, \bar{v} \subseteq \mathcal{U}$ and small $\mathcal{M} \preceq \mathcal{U}$:

(1) *Invariance under automorphisms of* $\mathcal{U}$.

(2) *Symmetry:* $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$.

(3) *Transitivity:* $A \mathop{\smile\!\!\!\!\!\!|}_C BD \iff A \mathop{\smile\!\!\!\!\!\!|}_C B \wedge A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$.

(4) *Finite character:* $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$ for every finite $\bar{a} \subseteq A$.

(5) *Existence:* There exists A' such that $\mathrm{tp}(A'/C) = \mathrm{tp}(A/C)$ and $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.

(6) *Local character:* There exists $B_0 \subseteq B$ such that $|B_0| \leq |T|$ and $\bar{u} \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.

(7) *Stationarity of types:* If $\mathrm{tp}(A/\mathcal{M}) = \mathrm{tp}(B/\mathcal{M})$ and $A \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} C$ and $B \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} C$, then
$\mathrm{tp}(A/\mathcal{M} \cup C) = \mathrm{tp}(B/\mathcal{M} \cup C)$.

Proposition 3.36 ([3]) *Let $\kappa > |T|$ and let $\mathcal{U}$ be a κ-universal domain. Then the theory T is stable if and only if there exists a stable independence relation on $\mathcal{U}$, and in this case the stable independence relation is the independence relation given by non-dividing.*

Thus, in order to prove that our theories are stable, we only need to define a stable independence relation. Ben Yaacov proved in [2, Theorem 4.1] that the classical relation of independence of events was the required one in the case of measure algebras without group actions (see also [5, Theorem 8.1]). Now that we described the definable closures in our theories, the proof of Ben Yaacov naturally adapts to this case.

Definition 3.37 From now on, we write $\langle\langle A\rangle\rangle$ for $\mathrm{dcl}^{\mathcal{U}}(A)$.

Let $A, B, C \subseteq \mathcal{U}$, we say that A and B are independent over C, and we write $A \mathop{\smile\!\!\!\!\!\!|}_C B$, if we have $\forall a \in \langle\langle A\rangle\rangle$, $\forall b \in \langle\langle B\rangle\rangle$, $\mathbb{P}_{\langle\langle C\rangle\rangle}(a)\mathbb{P}_{\langle\langle C\rangle\rangle}(b) = \mathbb{P}_{\langle\langle C\rangle\rangle}(a \wedge b)$.

We will need the following propositions:

Proposition 3.38 (Kallenberg [14, Proposition 5.6]) *Let $A, B, C \subseteq \mathcal{U} \models \mathfrak{A}_{F_\infty}$. Then we have $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $\forall a \in \langle\langle A\rangle\rangle$,*

$$\mathbb{P}_{\langle\langle BC\rangle\rangle}(a) = \mathbb{P}_{\langle\langle C\rangle\rangle}(a).$$

Proposition 3.39 ([2, Lemma 2.7]) *Let θ be a hyperfinite IRS on F_∞.*

Let $\mathcal{U} \models \mathfrak{A}'_\theta$ and let $\mathcal{M}_1, \mathcal{M}_2$ be small substructures of $\mathcal{U}$. Let $\mathcal{Z}$ be a common substructure of $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $\langle \mathcal{M}_1 \cup \mathcal{M}_2 \rangle$ be the substructure of $\mathcal{U}$ generated by $\mathcal{M}_1$ and $\mathcal{M}_2$ and define $\mathcal{N}$ the relative independent joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$ as in Theorem 3.30.

Then $\mathcal{M}_1 \mathop{\downarrow}_{\mathcal{Z}} \mathcal{M}_2$ if and only if $\langle \mathcal{M}_1 \cup \mathcal{M}_2 \rangle \simeq \mathcal{N}$.

Theorem 3.40 *If θ is a hyperfinite IRS, the relation of independence $\downarrow$ defined above is a stable independence relation when restricted to triples of small subsets, relatively to the theory $\mathfrak{A}_\theta$. Consequently, the theory $\mathfrak{A}_\theta$ is stable and the relation $\downarrow$ agrees with non-dividing on triples of small subsets.*

Proof

(1) *Invariance under automorphisms of $\mathcal{U}$:* If ρ is an automorphism of $\mathcal{U}$, by uniqueness of the orthogonal projection, we know that $\mathbb{P}_{\langle\langle \rho(C) \rangle\rangle} = \rho \circ \mathbb{P}_{\langle\langle C \rangle\rangle} \circ \rho^{-1}$. Therefore
$$\mathbb{P}_{\langle\langle C \rangle\rangle}(a)\mathbb{P}_{\langle\langle C \rangle\rangle}(b) = \mathbb{P}_{\langle\langle C \rangle\rangle}(a \wedge b)$$
if and only if
$$\mathbb{P}_{\langle\langle \rho(C) \rangle\rangle}(\rho a)\mathbb{P}_{\langle\langle \rho(C) \rangle\rangle}(\rho b) = \mathbb{P}_{\langle\langle \rho(C) \rangle\rangle}(\rho(a \wedge b)).$$

(2) *Symmetry:* The definition is symmetric.

(3) *Transitivity:* Let A, B, C, D be small. First if $A \mathop{\downarrow}_C B$ and $A \mathop{\downarrow}_{BC} D$ then by Proposition 3.38, for $a \in \langle\langle A \rangle\rangle$, we have $\mathbb{P}_{\langle\langle BCD \rangle\rangle}(a) = \mathbb{P}_{\langle\langle BC \rangle\rangle}(a) = \mathbb{P}_{\langle\langle C \rangle\rangle}(a)$ so $A \mathop{\downarrow}_C BD$.
Conversely, suppose that $A \mathop{\downarrow}_C BD$. Then $\mathbb{P}_{\langle\langle BCD \rangle\rangle}(a) = \mathbb{P}_{\langle\langle C \rangle\rangle}(a)$, but that implies that $\mathbb{P}_{\langle\langle C \rangle\rangle}(a)$ is a $\langle\langle C \rangle\rangle$-measurable function such that for all $\langle\langle BCD \rangle\rangle$-measurable function f we have $\int \mathbb{P}_{\langle\langle C \rangle\rangle}(a) f = \int \mathbb{1}_a f$. We conclude that $\mathbb{P}_{\langle\langle BCD \rangle\rangle}(a) = \mathbb{P}_{\langle\langle BC \rangle\rangle}(a) = \mathbb{P}_{\langle\langle C \rangle\rangle}(a)$, and therefore that $A \mathop{\downarrow}_C B$ and $A \mathop{\downarrow}_{BC} D$.

(4) *Finite character:* It follows from the definition and the continuity of $\mathbb{P}$.

(5) *Existence:* Let A, B, C be small subsets of $\mathcal{U}$. By the Löwenheim–Skolem theorem, let $\mathcal{A}$ and $\mathcal{B}$ be small structures such that $\langle\langle AC \rangle\rangle \subseteq \mathcal{A} \preceq \mathcal{U}$ and $\langle\langle BC \rangle\rangle \subseteq \mathcal{B} \preceq \mathcal{U}$, and let $\mathcal{C} = \langle\langle C \rangle\rangle$. Then $\mathcal{A}$ and $\mathcal{B}$ are both elementary substructures of $\mathcal{U}$ containing $\mathcal{I}_{\mathcal{U}}$. It follows that $\mathcal{A}$ and $\mathcal{B} \models \mathfrak{A}'_\theta$ when the constants S_γ are interpreted by supp $\gamma^{\mathcal{U}}$ in either of these models, and $\mathcal{C}$ is an $\mathcal{L}'_\infty$-common substructure of $\mathcal{A}$ and $\mathcal{B}$, so using (the proof of) Theorem 3.30,

we see that the relative independent joining $\mathcal{D}$ of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{C}$ is a small model of $\mathfrak{A}_\theta$.
By saturation and homogeneity of $\mathcal{U}$ and completeness of $\mathfrak{A}_\theta$, we can embed $\mathcal{D}$ in $\mathcal{U}$ while sending $\mathcal{B}$ back to $\mathcal{B}$. Taking the image of $\mathcal{A}$ by this embedding gives us a new copy $\mathcal{A}'$ of $\mathcal{A}$ and a new copy A' of A such that $\mathrm{tp}(A'/C) = \mathrm{tp}(A/C)$. Finally, $\langle \mathcal{A}' \cup \mathcal{B}\rangle \simeq \mathcal{D}$ so by Proposition 3.39 we get that $\mathcal{A}' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{C}} \mathcal{B}$, which in turn implies that $A' \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$.

(6) *Local character:* Let $\bar{u} = (u_1, \ldots, u_n) \subseteq \mathcal{U}$ be finite. Consider the conditional probabilities $\mathbb{P}_{\langle\langle B\rangle\rangle}(u_i)$. These are $\langle\langle B\rangle\rangle$-measurable functions with real values and so there is a countably generated σ-subalgebra of $\langle\langle B\rangle\rangle$, say $\langle\langle B_0\rangle\rangle$ where $B_0 \subseteq B$ is countable, for which they are all measurable. But then we have $\mathbb{P}_{\langle\langle B\rangle\rangle}(u_i) = \mathbb{P}_{\langle\langle B_0\rangle\rangle}(u_i)$, so by Proposition 3.38 $\bar{u} \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} B$.

(7) *Stationarity of types:* We denote by $\mathrm{tp}_{\mathcal{L}}(\bar{x}/Y)$ the type of a tuple $\bar{x}$ over a set of parameters Y in the language $\mathcal{L}$. In other words, this is the type of $\bar{x}$ over Y in the underlying atomless measure algebra of $\mathcal{U}$.
Let $A, B, C \subseteq \mathcal{U}$ be small and $\mathcal{M} \preceq \mathcal{U}$ be small. Suppose that $\mathrm{tp}(A/\mathcal{M}) = \mathrm{tp}(B/\mathcal{M})$, $A \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{M}} C$ and $B \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{M}} C$.
We begin by proving that $\mathrm{tp}_{\mathcal{L}}(A/\langle\langle \mathcal{M} \cup C\rangle\rangle) = \mathrm{tp}_{\mathcal{L}}(B/\langle\langle \mathcal{M} \cup C\rangle\rangle)$. Indeed, for $a \in \langle A\rangle$ and $b \in \langle B\rangle$, we have $\mathbb{P}_{\langle\langle \mathcal{M}\cup C\rangle\rangle}(a) = \mathbb{P}_{\mathcal{M}}(a)$ and $\mathbb{P}_{\langle\langle \mathcal{M}\cup C\rangle\rangle}(b) = \mathbb{P}_{\mathcal{M}}(b)$, but by Proposition 3.10 types in AMA can be fully described with conditional probabilities, and we have $\mathrm{tp}_{\mathcal{L}}(A/\mathcal{M}) = \mathrm{tp}_{\mathcal{L}}(B/\mathcal{M})$, so we get $\mathrm{tp}_{\mathcal{L}}(A/\langle\langle \mathcal{M} \cup C\rangle\rangle) = \mathrm{tp}_{\mathcal{L}}(B/\langle\langle \mathcal{M} \cup C\rangle\rangle)$.
Now Theorem 3.31 implies that $\mathrm{tp}(A/\mathcal{M} \cup C)$ is determined by the $\mathcal{L}$-type $\mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty A \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle \mathcal{M} \cup C\rangle\rangle\right)$, and that $\mathrm{tp}(B/\mathcal{M} \cup C)$ is determined by the $\mathcal{L}$-type $\mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty B \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle \mathcal{M} \cup C\rangle\rangle\right)$.
Thus, let $A' = F_\infty A \cup \mathcal{I}_{\mathcal{U}}$ and $B' = F_\infty B \cup \mathcal{I}_{\mathcal{U}}$.
It is clear that $\mathrm{tp}(A'/\mathcal{M}) = \mathrm{tp}(B'/\mathcal{M})$, $A' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{M}} C$ and $B' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{M}} C$, and we can apply what we proved just above to conclude that

$$\mathrm{tp}_{\mathcal{L}}\left(A'/\langle\langle \mathcal{M} \cup C\rangle\rangle\right) = \mathrm{tp}_{\mathcal{L}}\left(B'/\langle\langle \mathcal{M} \cup C\rangle\rangle\right),$$

that is

$$\mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty A \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle \mathcal{M} \cup C\rangle\rangle\right) = \mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty B \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle \mathcal{M} \cup C\rangle\rangle\right),$$

hence the conclusion. □

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