



# Generalized effective completeness for continuous logic

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*Abstract:* In this paper, we present a generalized effective completeness theorem for continuous logic. The primary result is that any continuous theory is satisfied in a metric structure which admits a presentation that is Turing reducible to that theory. It then follows that any decidable theory is satisfied by a computably presentable metric structure. This work runs parallel to the effective completeness results of Didehvar, Ghasemloo, and Pourmahdian, as well as those of Calvert, though given in the setting of computable presentations.

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## 1 Introduction

Completeness results relate theories to structures. *Effective* completeness results relate *decidable* theories to *computable* structures. The first such result was given by Millar in [10]. However, the method provided in that manuscript only applies to classical logic and classically computable structures. Hence, it cannot be directly applied to continuous logic and uncountable structures.

In [1], Ben Yaacov et al developed a model theory for metric structures using continuous first-order logic, and a completeness result was proven by Ben Yaacov and Pedersen in [2]. Calvert then extended this result to an effective version of completeness, relating decidable theories in continuous logic to probabilistically decidable structures.

**Theorem 1** ([5, Theorem 4.5]) *Let  $T$  be a complete, decidable, continuous first-order theory. Then there is a probabilistically decidable, continuous weak structure  $\mathfrak{M}$  such that  $\mathfrak{M} \models T$ .*

In the last decade, however, computable presentations rather than probabilistic decidability have become standard for the study of effectivity on metric structures (see, eg,

Brown, McNicholl, and Melnikov [4] and Franklin and McNicholl [8]). The motivation for this paper was, therefore, the question “Is there an effective completeness theorem for continuous logic and *computable presentations*?”

Didehvar, Ghasemloo, and Pourmahdian provided one version of an answer to this question in [7], perhaps implicitly with respect to computable presentations. The result proven was a qualified effective completeness result for the first-order rational Pavelka logic ( $\mathbf{RPL}\forall$ ).

**Theorem 2** ([7, Theorem 3.5]) *Every consistent, linear-complete, computably axiomatizable Henkin theory in  $\mathbf{RPL}\forall$  has a decidable model.*

As the authors note, the above theorem also applies to the continuous logic of [1], since it is a fragment of  $\mathbf{RPL}\forall$ .

Our primary result can then be considered as running parallel to these results, while being firmly situated in the setting of computable presentations. We show that there is an effective procedure which, given a name of a continuous theory, produces a presentation of a metric structure which models that theory. In other words, any continuous theory is satisfied in a structure which admits a presentation that is Turing reducible to that theory. Hence, if the theory is decidable, the presentation is computable. Of note is that the effective procedure constructs a presentation of a bona fide metric structure, rather than a weak structure, as given by Calvert.

In Section 2, we provide a brief review of continuous logic, metric structures, computable analysis, and computable presentations. Section 3 recalls previous results in the model theory of metric structures. These results are then extended to important preliminary model-theoretic propositions in Section 4.1. Section 4.2 follows to include our primary lemma, allowing us to uniformly effectively extend theories to complete theories. We then prove our main theorem, a generalized version of effective completeness, in Section 4.3. Standard effective completeness follows from this as the concluding corollary.

## 2 Background

### 2.1 Continuous logic

The *logical symbols* of continuous logic differ from those of classical logic primarily in the sense that  $\neg$ ,  $\frac{1}{2}$ , and  $\dot{\div}$  are the *connectives*, while  $\sup$  and  $\inf$  serve as the

quantifiers.<sup>1</sup> A signature is a quintuple  $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \Delta, \eta)$  where  $\mathcal{P}$ ,  $\mathcal{F}$ , and  $\mathcal{C}$  are mutually disjoint sets of predicate, function, and constant symbols,  $\Delta : \mathcal{P} \cup \mathcal{F} \rightarrow \mathbb{N}^{\mathbb{N}}$  is a modulus map,  $\eta : \mathcal{P} \cup \mathcal{F} \rightarrow \mathbb{N} \setminus \{0\}$  is an arity map, and there is a distinguished binary predicate symbol  $\underline{d} \in \mathcal{P}$  such that  $\Delta(\underline{d}) = \text{id}_{\mathbb{N}}$ , which ultimately will be interpreted as a pseudometric or metric. For the remainder of this section, unless stated otherwise, we will assume we have a fixed signature  $L$ .

The construction of terms and well-formed formulas (wffs) is straightforward, as are the definitions of free variables and sentences, though explicit definitions can be found in [1]. Moreover, the following syntax maps are used as shorthand.

Shorthand	String
$\varphi \vee \psi$	$\neg((\neg\varphi) \dot{\div} \psi)$
$\varphi \wedge \psi$	$\varphi \dot{\div} (\varphi \dot{\div} \psi)$
$\underline{0}$	$\sup_x \underline{d}(x, x)$
$\underline{1}$	$\neg \underline{0}$
$\varphi \dot{+} \psi$	$\neg((\underline{1} \dot{\div} \varphi) \dot{\div} \psi)$
$m\varphi$	$\underbrace{(\dots(\varphi \dot{+} \varphi) \dot{+} \dots \dot{+} \varphi)}_{m\text{-many}}$
$\underline{2^{-k}}$	$\underbrace{\frac{1}{2} \dots \frac{1}{2}}_{k\text{-many}} \underline{1}$
$\frac{\ell}{2^k}$	$\underbrace{(\dots(\underline{2^{-k}} \dot{+} \underline{2^{-k}}) \dot{+} \dots \dot{+} \underline{2^{-k}})}_{\ell\text{-many}}$

It is also important to note, perhaps prematurely, some semantic heuristics. In contrast to classical logic,  $\dot{\div}$  can be interpreted as “only if”,  $\vee$  as “and”,  $\wedge$  as “or”, and each  $\frac{\ell}{2^k}$  as the dyadic number  $\frac{\ell}{2^k}$ .

<sup>1</sup> In some versions of continuous logic, the set of connectives contains a distinguished symbol  $\underline{u}$  for each continuous map  $u : [0, 1]^{\eta(u)} \rightarrow [0, 1]$ . The resulting set of well-formed formulas for such a logic is, however, uncountable and thus fails to perform effectively. Our choice of  $\neg$ ,  $\frac{1}{2}$ , and  $\dot{\div}$  as the connectives was made for four reasons. First,  $\neg$  plays precisely the role of classical negation ( $\neg$ ) and  $\dot{\div}$  of reverse implication ( $\leftarrow$ ). The interpretation of the  $\frac{1}{2}$  operator is similarly intuitive, as will be shown in the following subsection. Second, in Ben Yaacov and Usvyatsov [3], it was shown that after interpretation,  $\neg$ ,  $\frac{1}{2}$ , and  $\dot{\div}$  are dense in the set of all continuous maps on  $[0, 1]$ . Thus finitary well-formed formulas in these connectives can approximate those in the wider set of connectives arbitrarily well. Such an approximation is, moreover, sufficient for completeness (as seen in Ben Yaacov and Pedersen [2]). Third, when a signature is effectively numbered, the sentences and well-formed formulas of that signature may be effectively enumerated. And finally, due to the previous remarks,  $\neg$ ,  $\frac{1}{2}$ , and  $\dot{\div}$  have become a somewhat canonical set of connectives for continuous logic.

The standard list of axiom schemata for continuous logic can be found in [2], and a simplified list, more parsimonious for effective constructions, is given in Camrud [6]. The rules of inference can also be found in those sources.

## 2.2 Metric structures

Continuous logic was developed with the purpose of describing pseudometric and metric structures. On this note, a signature must be able to speak about the continuity of maps on such structures.

**Definition 1** Let  $(|\mathfrak{M}|, d)$  and  $(|\mathfrak{M}'|, d')$  be pseudometric spaces of diameter 1 and let  $f : |\mathfrak{M}| \rightarrow |\mathfrak{M}'|$ . A map  $\Delta(f) : \mathbb{N} \rightarrow \mathbb{N}$  is called a *modulus of continuity* for  $f$  if for every  $a, b \in |\mathfrak{M}|$ ,  $d(a, b) < 2^{-\Delta(f;n)}$  implies that  $d'(f(a), f(b)) \leq 2^{-n}$ .

An *interpretation* of  $L$  is a map  $\cdot^{\mathfrak{M}}$  with domain  $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$  such that for some *universe*  $|\mathfrak{M}|$ , each of the following hold.

- For every predicate symbol  $P$ ,  $P^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(P)} \rightarrow [0, 1]$ .
- For every function symbol  $f$ ,  $f^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(f)} \rightarrow |\mathfrak{M}|$ .
- For every constant symbol  $c$ ,  $c^{\mathfrak{M}} \in |\mathfrak{M}|$ .

$\cdot^{\mathfrak{M}}$  is a *continuous interpretation* if, moreover, each of the following hold.

- $\underline{d}^{\mathfrak{M}} := d$  is a pseudometric.
- For every predicate symbol  $P$ ,  $\Delta(P)$  is a modulus of continuity for  $P$ .<sup>2</sup>
- For every function symbol  $f$ ,  $\Delta(f)$  is a modulus of continuity for  $f$ .

When  $\cdot^{\mathfrak{M}}$  is an interpretation, the quintuple

$$\mathfrak{M} = (|\mathfrak{M}|, d, \{P^{\mathfrak{M}} : P \in \mathcal{P} \setminus \{\underline{d}\}\}, \{f^{\mathfrak{M}} : f \in \mathcal{F}\}, \{c^{\mathfrak{M}} : c \in \mathcal{C}\})$$

is an  $L$ -pre-structure. Moreover, if  $\cdot^{\mathfrak{M}}$  is a continuous interpretation,  $\mathfrak{M}$  is a *continuous*  $L$ -pre-structure. Lastly, if  $\cdot^{\mathfrak{M}}$  is a continuous interpretation and  $(|\mathfrak{M}|, d)$  is a complete metric space, then  $\mathfrak{M}$  is an  $L$ -structure. If  $|\mathfrak{M}|$  is countable,  $\mathfrak{M}$  is called *weak*.

When  $\mathfrak{M}$  is an  $L$ -pre-structure,  $\mathcal{P}^{\mathfrak{M}}$  is the set of *predicates* of  $\mathfrak{M}$ ,  $\mathcal{F}^{\mathfrak{M}}$  the set of *functions* of  $\mathfrak{M}$ , and  $\mathcal{C}^{\mathfrak{M}}$  the set of *distinguished points* of  $\mathfrak{M}$ .

For our purposes, the language of *non*-continuous pre-structures is dropped, and every pre-structure is assumed to be continuous. Also what were given here as “ $L$ -structures”

<sup>2</sup>Here the domain of  $P^{\mathfrak{M}}$  is considered as the pseudometric space  $(|\mathfrak{M}|^{\eta(P)}, (\underline{d}^{\mathfrak{M}})^{\eta(P)})$  and the range the metric space  $([0, 1], |\cdot|)$ .

are often designated as “metric  $L$ -structures”. In this manuscript, however, we assume every structure is interpreting a continuous signature, so we drop the prefix “metric”.

Variable assignments ( $\sigma$  or, for more specificity,  $\sigma(x \mapsto a)$ ) and interpretations ( $\cdot^{\mathfrak{M}, \sigma}$ ) of terms in a given structure are defined naturally, though a precise description can be found in [2]. Finally, the *value* of a well-formed formula  $\varphi$  is defined recursively as follows.

- $(P(t_0, \dots, t_n))^{\mathfrak{M}, \sigma} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}, \sigma}, \dots, t_n^{\mathfrak{M}, \sigma})$ .
- $(\neg \varphi)^{\mathfrak{M}, \sigma} := 1 - \varphi^{\mathfrak{M}, \sigma}$ .
- $(\frac{1}{2} \varphi)^{\mathfrak{M}, \sigma} := \frac{1}{2} \cdot \varphi^{\mathfrak{M}, \sigma}$ .
- $(\varphi \dot{-} \psi)^{\mathfrak{M}, \sigma} := \max \{ \varphi^{\mathfrak{M}, \sigma} - \psi^{\mathfrak{M}, \sigma}, 0 \}$ .
- $(\sup_x \varphi)^{\mathfrak{M}, \sigma} := \sup_{a \in |\mathfrak{M}|} \varphi^{\mathfrak{M}, \sigma(x \mapsto a)}$ .
- $(\inf_x \varphi)^{\mathfrak{M}, \sigma} := \inf_{a \in |\mathfrak{M}|} \varphi^{\mathfrak{M}, \sigma(x \mapsto a)}$ .

When  $\varphi^{\mathfrak{M}, \sigma} = 0$ ,  $\mathfrak{M}$  with  $\sigma$  satisfies  $\varphi$  ( $\mathfrak{M}, \sigma \models \varphi$ ). That 0 represents satisfaction differs importantly from classical logic since the predicate  $d$  plays the role of equality, ie, when  $d$  is a metric,  $d(a, b) = 0$  if and only if  $a = b$ .

Sometimes, one structure may embed into another of the same signature. When  $L$  is that signature, this is called an  $L$ -morphism or  $L$ -embedding, and is defined in detail in Definition 6.5 of [2].

### 2.3 Computable analysis and presentations

Computable analysis is summarized well in Weihrauch [11]. For our purposes, however, we need only mention the definitions of a computable real number and a computable map into the reals.

**Definition 2** A real number  $r$  is *computable* if there is an effective procedure which, given  $k \in \mathbb{N}$ , outputs a rational  $q \in \mathbb{Q}$  such that

$$|r - q| < 2^{-k}.$$

When  $A$  is a countable set, a map  $f : A \rightarrow \mathbb{R}$  is *computable* if there is an effective procedure which, given  $a \in A$  and  $k \in \mathbb{N}$ , outputs a rational  $q \in \mathbb{Q}$  such that

$$|f(a) - q| < 2^{-k}.$$

Since metric structures often have uncountable domains, a method for discussing effectivity on such structures was introduced by Melnikov [9], recently also seen in Franklin and McNicholl [8]. An effectively numbered signature is necessary for this method.

**Definition 3** A signature  $L$  is *effectively numbered* if there is an effective mapping of the natural numbers onto  $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$  and, moreover, an effective procedure which, given the code of a predicate or function symbol, outputs that symbol's arity and an index of a Turing machine which serves as a modulus of continuity for that symbol.

We now introduce computable presentations. From here we will assume we are working under a fixed effectively numbered signature  $L$ .

**Definition 4** Given an  $L$ -structure  $\mathfrak{M}$  and  $A \subseteq |\mathfrak{M}|$ , the *algebra generated by  $A$*  is the smallest subset of  $|\mathfrak{M}|$  containing  $A$  that is closed under every function of  $\mathfrak{M}$ .

A pair  $(\mathfrak{M}, g)$  is called a *presentation* of  $\mathfrak{M}$  if  $g : \mathbb{N} \rightarrow |\mathfrak{M}|$  is a map such that the algebra generated by  $\text{ran}(g)$  is dense. Such an  $(\mathfrak{M}, g)$  is also denoted as  $\mathfrak{M}^\sharp$ . Every point in  $\text{ran}(g)$  is called a *distinguished point* of the presentation, and each point in the algebra generated by the distinguished points is called a *rational point* of the presentation ( $\mathbb{Q}(\mathfrak{M}^\sharp)$ ). Notably,  $\text{ran}(g)$  does not need to be dense, but the *algebra* it generates does.

**Definition 5** A presentation  $\mathfrak{M}^\sharp$  is *computable* if the predicates of  $\mathfrak{M}$  are uniformly computable on the rational points of  $\mathfrak{M}^\sharp$ .<sup>3</sup>

An *index* of a computable presentation  $\mathfrak{M}^\sharp$  is an index of a Turing machine which, given a code of  $P \in \mathcal{P}$ , codes of  $a_0, \dots, a_{\eta(P)-1} \in \mathbb{Q}(\mathfrak{M}^\sharp)$ , and  $k \in \mathbb{N}$ , outputs a code of a rational  $q$  such that

$$|P^{\mathfrak{M}}(a_0, \dots, a_{\eta(P)-1}) - q| < 2^{-k}.$$

**Example 1** Let  $\mathfrak{M}$  be the metric structure consisting of the continuum  $[0, 1]$ , the Euclidean metric, and as functions the average map

$$\text{avg}(x, y) = \frac{x + y}{2}$$

and bounded addition

$$\dagger(x, y) = \min\{x + y, 1\}.$$

Defining  $g : \mathbb{N} \rightarrow [0, 1]$  as  $g(n) := n \bmod 2$  provides a computable presentation. While  $\text{ran}(g)$  is only  $\{0, 1\}$ , the algebra generated by  $\text{ran}(g)$  is all dyadics in  $[0, 1]$ , which is clearly dense in  $[0, 1]$ . Moreover, the Euclidean metric is uniformly computable on those dyadics.

<sup>3</sup>Since the metric is a binary predicate on  $\mathfrak{M}$ , this entails that the distance between any two rational points is uniformly computable.

### 3 Previous completeness results

We now recall many results related to completeness which were proven in [2]. We have altered some of the notation in order to make these results more applicable to our work, but the results proven remain the same. Important to this work is the introduction of the formal notion of dyadic numerals.

**Definition 6** The *dyadic numerals* (Dyad) are all sentences of the form  $\frac{\ell}{2^k}$  for  $\ell, k \in \mathbb{N}$ . When  $\underline{p} \in \text{Dyad}$ , by  $p$  we mean the real number such that for every  $L$ -pre-structure  $\mathfrak{M}$ ,  $p^{\mathfrak{M}} = p$ . Because of this, when no confusion is likely, we will simply write  $p$  for  $\underline{p}$ .

Maximal consistency is defined similarly to the classical case, but with an extra condition concerning limiting behavior.

**Definition 7** A set of wffs  $\Gamma$  is *maximally consistent* if for every pair of wffs  $\varphi$  and  $\psi$ , the following hold.

- (i) If  $\Gamma \vdash \varphi \dot{-} 2^{-k}$  for every  $k \in \mathbb{N}$ , then  $\varphi \in \Gamma$ .
- (ii) Either  $\varphi \dot{-} \psi \in \Gamma$  or  $\psi \dot{-} \varphi \in \Gamma$ .

Notably, without condition (i), we would not gain the intuitive property that if  $\Gamma$  is maximally consistent, then for every  $\varphi \notin \Gamma$ ,  $\Gamma \cup \{\varphi\}$  is inconsistent.

In [2], Ben Yaacov and Pedersen implemented a continuous version of a Henkin construction to prove their completeness theorem. To accomplish this, Henkin witnesses must be added to a signature.

**Definition 8** Given a signature  $L$ , the *Henkin extended signature of  $L$*  ( $L^+$ ) is the smallest signature that extends  $L$  and that, for every combination of  $L^+$ -wff  $\varphi$ , variable symbol  $x$ , and  $p, q \in \text{Dyad}$ , contains a unique constant symbol  $c_{\varphi, x, p, q}$ .

When  $\Gamma$  is a set of  $L^+$ -wffs, we say it is *Henkin complete* if for every  $L^+$ -wff  $\varphi$ , every variable symbol  $x$ , and every  $p, q \in \text{Dyad}$ ,

$$\left( \sup_x \varphi \dot{-} q \right) \wedge \left( p \dot{-} \varphi[c_{\varphi, x, p, q}/x] \right) \in \Gamma.$$

We now note a relevant lemma and theorem from Ben Yaacov and Pedersen.

**Lemma 1** ((ii) of [2, Lemma 8.5]) *Let  $T$  be an  $L$ -theory. Then for every pair of  $L$ -wffs  $\varphi$  and  $\psi$ , either  $T \cup \{\varphi \dot{-} \psi\}$  or  $T \cup \{\psi \dot{-} \varphi\}$  is consistent.*

**Theorem 3** (From [2, Theorem 8.10 and Proposition 9.2]) *Let  $T$  be an  $L$ -theory. Then there exists a maximally consistent, Henkin complete set of  $L^+$ -wffs  $\Gamma$  which extends  $T$ .*

In what follows, the original Henkin model created will be a continuous  $L^+$ -pre-structure. To make the move to a genuine  $L^+$ -structure, the following theorem is needed.

**Theorem 4** ([2, Theorem 6.9]) *Let  $\mathfrak{M}'$  be a continuous  $L$ -pre-structure. Then there is an  $L$ -structure  $\mathfrak{M}$  and an elementary  $L$ -embedding of  $\mathfrak{M}'$  into  $\mathfrak{M}$ .*

We now summarize the construction of the Henkin model from [2]. Completeness follows.

**Definition 9** Let  $\Gamma$  be a maximally consistent, Henkin complete set of  $L^+$ -wffs. Define the *Henkin continuous  $L^+$ -pre-structure over  $\Gamma$*  ( $\mathfrak{M}'_\Gamma$ ) as follows.

- $|\mathfrak{M}'_\Gamma|$  is the set of all terms of  $L^+$ .
- For every constant symbol  $c$  of  $L^+$ ,  $c^{\mathfrak{M}'_\Gamma} := c$ .
- For every function symbol  $f$  of  $L^+$  and  $t_0, \dots, t_{\eta(f)-1} \in |\mathfrak{M}'_\Gamma|$ ,

$$f^{\mathfrak{M}'_\Gamma}(t_0, \dots, t_{\eta(f)-1}) := f(t_0, \dots, t_{\eta(f)-1}).$$

- For every predicate symbol  $P$  of  $L^+$  and  $t_0, \dots, t_{\eta(P)-1} \in |\mathfrak{M}'_\Gamma|$ ,

$$P^{\mathfrak{M}'_\Gamma}(t_0, \dots, t_{\eta(P)-1}) := \sup \{p \in [0, 1] : p \in \text{Dyad and } p \div P(t_0, \dots, t_{\eta(P)-1}) \in \Gamma\}.$$

The *basic assignment* on  $\mathfrak{M}'_\Gamma$  is defined as  $\sigma(x) := x$  for every variable symbol  $x$  of  $L^+$ . By a slight abuse of notation, when  $\mathfrak{M}'_\Gamma$  is a Henkin continuous  $L^+$ -pre-structure, by  $\varphi^{\mathfrak{M}'_\Gamma}$  we mean  $\varphi^{\mathfrak{M}'_\Gamma, \sigma}$ , and by  $\mathfrak{M}'_\Gamma \models \varphi$  we mean  $\mathfrak{M}'_\Gamma, \sigma \models \varphi$ , where  $\sigma$  is the basic assignment. The *Henkin  $L^+$ -structure over  $\Gamma$*  ( $\mathfrak{M}_\Gamma$ ) is the structure induced by the metric completion of  $(|\mathfrak{M}'_\Gamma|, \underline{d}^{\mathfrak{M}'_\Gamma})$  and the elementary morphism given in Theorem 4.

**Theorem 5** ([2, Theorem 9.4]) *Let  $\Gamma$  be a maximally consistent, Henkin complete set of  $L^+$ -wffs. Then  $\mathfrak{M}_\Gamma \models \Gamma$ .*

**Corollary 1** (Completeness of Continuous Logic [2, Theorem 9.5]) *A set of  $L$ -wffs is consistent if and only if it is completely satisfiable.*

Ben Yaacov and Pedersen then introduce important maps from sets of  $L$ -wffs into  $[0, 1]$ . These maps serve as upper bounds on relative provability and interpretation of sentences following from those sets of  $L$ -wffs.



**Definition 10** Let  $\Gamma$  be a set of  $L$ -wffs. The *degree of truth with respect to  $\Gamma$*  ( $\cdot \overset{\circ}{\Gamma}$ ) is a map from wffs to  $[0, 1]$ , defined as

$$\varphi_{\Gamma}^{\circ} := \sup \{ \varphi^{\mathfrak{M}, \sigma} : \mathfrak{M}, \sigma \models \Gamma \}.$$

The *degree of provability with respect to  $\Gamma$*  ( $\cdot \overset{\odot}{\Gamma}$ ) is a similar map, defined as

$$\varphi_{\Gamma}^{\odot} := \inf \{ p \in [0, 1] : p \in \text{Dyad and } \Gamma \vdash \varphi \dot{-} p \}.$$

The Completeness Theorem then implies that these maps are the same.

**Corollary 2** ([2, Corollary 9.8]) *For any  $L^+$ -wff  $\varphi$  and set of  $L$ -wffs  $\Gamma$ ,  $\varphi_{\Gamma}^{\circ} = \varphi_{\Gamma}^{\odot}$ .*

**Definition 11** A set of  $L$ -wffs  $\Gamma$  is *complete* if there is a structure  $\mathfrak{M}$  and assignment  $\sigma$  such that for every  $L$ -wff  $\varphi$ ,

$$\varphi_{\Gamma}^{\circ} = \varphi^{\mathfrak{M}, \sigma}.$$

In contrast to the classical case, even if a theory is complete, its set of consequences may not be maximally consistent. This is due to the limiting behavior condition discussed in Definition 7. The Deduction Theorem for continuous logic encounters a similar issue.

**Theorem 6** (Deduction Theorem [2, Theorem 8.1]) *Let  $\Gamma$  be a set of  $L$ -wffs. Then for every  $L$ -wff  $\psi$ ,  $\Gamma \cup \{ \psi \} \vdash \varphi$  if and only if  $\Gamma \vdash \varphi \dot{-} m\psi$ , for some  $m \in \mathbb{N}$ .*

We also note the Generalization Theorem, which will be useful in what follows.

**Lemma 2** (Generalization Theorem [2, Lemma 8.2]) *Let  $\Gamma$  be a set of  $L^+$ -wffs and  $\varphi$  an  $L^+$ -wff. If  $x$  does not appear freely in  $\Gamma$  and  $\Gamma \vdash \varphi$ , then  $\Gamma \vdash \sup_x \varphi$ .*

And lastly, we note the following lemma of Calvert's.

**Lemma 3** ([5, Lemma 4.6]) *There is an effective procedure which extends  $L$  to its Henkin extended signature  $L^+$ .*

## 4 Main result

### 4.1 Model-theoretic preliminaries

In this section, we prove four important model-theoretic propositions which extend the results of [2] and are vital to the proof of the generalized effective completeness theorem.

**Proposition 1** Let  $\Gamma$  be a set of  $L$ -wffs and  $B$  a finite set of  $L$ -wffs. Then for every  $L$ -wff  $\varphi$ :

$$\varphi_{\Gamma \cup B}^{\circ} \leq \left( \varphi \dot{\div} \bigvee_{\theta \in B} \theta \right)_{\Gamma}^{\circ}$$

**Proof** Fix an  $L$ -wff  $\varphi$ . If  $\varphi_{\Gamma \cup B}^{\circ} = 0$  the result follows trivially. Thus, suppose  $\varphi_{\Gamma \cup B}^{\circ} > 0$ . Notably, this implies that  $\Gamma \cup B$  is consistent. Fix  $p \in \text{Dyad}$  such that  $p < \varphi_{\Gamma \cup B}^{\circ}$ . Then there is some  $L$ -structure  $\mathfrak{M}$  and assignment  $\sigma$  such that  $\mathfrak{M}, \sigma \models \Gamma \cup B$  while  $\varphi^{\mathfrak{M}, \sigma} > p$ . But, clearly, since  $\mathfrak{M}, \sigma \models B$ ,  $\left( \bigvee_{\theta \in B} \theta \right)^{\mathfrak{M}, \sigma} = 0$ . Hence,  $\left( \varphi \dot{\div} \bigvee_{\theta \in B} \theta \right)^{\mathfrak{M}, \sigma} > p$ . Then since  $\mathfrak{M}, \sigma \models \Gamma$ , this implies that  $\left( \varphi \dot{\div} \bigvee_{\theta \in B} \theta \right)_{\Gamma}^{\circ} > p$ . Since this is true for every  $p < \varphi_{\Gamma \cup B}^{\circ}$ , we have that  $\varphi_{\Gamma \cup B}^{\circ} \leq \left( \varphi \dot{\div} \bigvee_{\theta \in B} \theta \right)_{\Gamma}^{\circ}$ .  $\square$

**Proposition 2** Let  $\Gamma$  be a set of  $L$ -wffs and  $B$  a finite set of  $L$ -wffs such that  $\Gamma \cup B$  is consistent. Then there are infinitely many  $L$ -wffs  $\varphi$  such that

$$\left( \varphi \dot{\div} \bigvee_{\theta \in B} \theta \right)_{\Gamma}^{\circ} = 1.$$

**Proof** Recall that since  $\Gamma \cup B$  is consistent, we may fix some  $L$ -wff  $\varphi$  such that  $\Gamma \cup B \not\vdash \varphi$ . By Corollary 2,  $\varphi_{\Gamma \cup B}^{\circ} > \frac{1}{M}$ , for some  $M \in \mathbb{N}$ . It follows that for every  $m \geq M$ ,  $(m\varphi)_{\Gamma \cup B}^{\circ} = 1$ . Hence, by Proposition 1,  $\left( m\varphi \dot{\div} \bigvee_{\theta \in B} \theta \right)_{\Gamma}^{\circ} = 1$ , for every  $m \geq M$ .  $\square$

Note that for our purposes, a theory is a consistent set of *sentences*. Therefore, theories do not contain any free variables.

**Proposition 3** Let  $T$  be an  $L$ -theory and  $\varphi$  an  $L$ -wff with free variables  $\vec{x}$ . Then:

$$\varphi_T^{\circ} = \left( \sup_{\vec{x}} \varphi \right)_T^{\circ}$$

**Proof** Fix a signature  $L$ , an  $L$ -theory  $T$ , and an  $L$ -wff  $\varphi$  with free variables  $\vec{x}$ . Notice that since  $T$  contains only  $L$ -sentences, none of  $\vec{x}$  appear freely in  $T$ . It follows via Corollary 2 and the Generalization Theorem that:

$$\begin{aligned} \varphi_T^{\circ} &= \inf \{ p : p \in \text{Dyad} \text{ and } T \vdash \varphi \dot{\div} p \} \\ &= \inf \{ p : p \in \text{Dyad} \text{ and } T \vdash \sup_{\vec{x}} \varphi \dot{\div} p \} \\ &= \left( \sup_{\vec{x}} \varphi \right)_T^{\circ} \end{aligned} \quad \square$$

**Proposition 4** Let  $T$  be an  $L$ -theory. Then for every  $L^+$ -wff  $\theta$ ,

$$\left(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\right)_T^\circ = \theta_T^\circ$$

where  $\vec{c}$  is the tuple of all constants from  $L^+$ , but not in  $L$ , appearing in  $\theta$ .

**Proof** Fix an  $L$ -theory  $T$  and an  $L^+$ -wff  $\theta$ . Recall that no variable from  $\vec{x}$  appears freely in  $T$ , nor does any constant in  $\vec{c}$  appear in  $T$ , since it is an  $L$ -theory. Hence for every  $L$ -structure  $\mathfrak{M}$  such that  $\mathfrak{M} \models T$ , and every tuple  $\vec{a} \in |\mathfrak{M}|$ , there is an  $L^+$ -structure  $\mathfrak{M}_a^+$  such that  $\vec{c}^{\mathfrak{M}_a^+} = \vec{a}$  and  $\mathfrak{M}_a^+ \upharpoonright_L = \mathfrak{M}$ . Then:

$$\begin{aligned} \left(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\right)_T^\circ &= \sup \left\{ \left(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\right)^{\mathfrak{M}} : \mathfrak{M} \models T \right\} \\ &= \sup \left\{ \sup_{\{\sigma(\vec{x} \mapsto \vec{a}) : \vec{a} \in |\mathfrak{M}|\}} (\theta[\vec{x}/\vec{c}])^{\mathfrak{M}, \sigma(\vec{x} \mapsto \vec{a})} : \mathfrak{M} \models T \right\} \\ &= \sup \left\{ (\theta[\vec{x}/\vec{c}])^{\mathfrak{M}, \sigma(\vec{x} \mapsto \vec{a})} : \mathfrak{M}, \sigma(\vec{x} \mapsto \vec{a}) \models T, \vec{a} \in |\mathfrak{M}| \right\} \\ &\leq \sup \left\{ (\theta[\vec{x}/\vec{c}])^{\mathfrak{M}_a^+, \sigma(\vec{x} \mapsto \vec{a})} : \mathfrak{M}_a^+, \sigma(\vec{x} \mapsto \vec{a}) \models T, \vec{a} \in |\mathfrak{M}| \right\} \\ &= \sup \left\{ \theta^{\mathfrak{M}_a^+, \sigma} : \mathfrak{M}_a^+, \sigma \models T \right\} \\ &\leq \sup \left\{ \theta^{\mathfrak{M}^+, \sigma} : \mathfrak{M}^+, \sigma \models T \right\} \\ &= \theta_T^\circ \end{aligned}$$

Now notice that for any  $L^+$ -structure  $\mathfrak{M}^+$  and assignment  $\sigma$ , there is an assignment  $\sigma(\vec{x} \mapsto \vec{c}^{\mathfrak{M}^+})$ . Then the  $L$ -structure  $\mathfrak{M}^+ \upharpoonright_L$  is such that  $\theta^{\mathfrak{M}^+, \sigma} = (\theta[\vec{x}/\vec{c}])^{\mathfrak{M}^+ \upharpoonright_L, \sigma(\vec{x} \mapsto \vec{c}^{\mathfrak{M}^+})}$ .

It follows that:

$$\begin{aligned} \theta_T^\circ &= \sup \left\{ \theta^{\mathfrak{M}^+, \sigma} : \mathfrak{M}^+, \sigma \models T \right\} \\ &= \sup \left\{ (\theta[\vec{x}/\vec{c}])^{\mathfrak{M}^+ \upharpoonright_L, \sigma(\vec{x} \mapsto \vec{c}^{\mathfrak{M}^+})} : \mathfrak{M}^+ \upharpoonright_L, \sigma(\vec{x} \mapsto \vec{c}^{\mathfrak{M}^+}) \models T \right\} \\ &\leq \sup \left\{ (\theta[\vec{x}/\vec{c}])^{\mathfrak{M}, \sigma} : \mathfrak{M}, \sigma \models T \right\} \\ &= (\theta[\vec{x}/\vec{c}])_T^\circ \end{aligned}$$

But by Proposition 3,  $(\theta[\vec{x}/\vec{c}])_T^\circ = (\sup_{\vec{x}} \theta[\vec{x}/\vec{c}])_T^\circ$ . Therefore,  $\theta_T^\circ \leq (\sup_{\vec{x}} \theta[\vec{x}/\vec{c}])_T^\circ$ . The claim follows.  $\square$

## 4.2 Effectively extending theories

Since any  $L$ -theory  $T$  has an associated degree of truth map  $\cdot_T^\circ$ , to analyze the effectiveness of a theory we will actually consider the effectiveness of the related degree of truth map. The following definition was given by Ben Yaacov and Pedersen in [2].

**Definition 12** An  $L$ -theory  $T$  is *decidable* if  $\cdot_T^\circ$  is a computable map from the set of wffs to  $[0, 1]$ .

Since there are uncountably-many such maps, we introduce a naming system.

**Definition 13** Given an  $L$ -theory  $T$ , we say that  $X \in \mathbb{N}^{\mathbb{N}}$  is a *name* of  $T$  if the following hold.

- For every  $n, k \in \mathbb{N}$ , there is some  $m \in \mathbb{N}$  such that  $\langle n, k, m \rangle \in \text{ran}(X)$ .
- For every  $n, k, m \in \mathbb{N}$ , if  $\langle n, k, m \rangle \in \text{ran}(X)$ , then  $q_m \in [(\varphi_n)_T^\circ - 2^{-k}, (\varphi_n)_T^\circ + 2^{-k}]$ .

**Proposition 5** An  $L$ -theory is decidable if and only if it has a computable name.

**Proof** For the forward direction, suppose  $T$  is a decidable  $L$ -theory. Recall that this means  $\cdot_T^\circ$  is computable, ie there is an effective procedure which, given the pair  $n$  and  $k$  as inputs, outputs  $m$  such that  $q_m \in [(\varphi_n)_T^\circ - 2^{-k}, (\varphi_n)_T^\circ + 2^{-k}]$ . Fix one such procedure. For every code of a pair, define  $X(\langle n, k \rangle) := \langle n, k, m \rangle$  where that procedure outputs  $q_m$  when given  $n$  and  $k$ . On every natural number that fails code a pair, let  $X$  be 0. By construction  $X \in \mathbb{N}^{\mathbb{N}}$  is computable, and, by simple inspection, a name of  $T$ .

For the reverse direction, suppose  $X$  is a computable name of an  $L$ -theory  $T$ . We construct an effective procedure which will witness the computability of  $\cdot_T^\circ$  as follows. Given the pair  $n$  and  $k$ , begin computing  $\text{ran}(X)$  until a code of a triple of the form  $\langle n, k, m \rangle$  is output. It follows that  $q_m \in [(\varphi_n)_T^\circ - 2^{-k}, (\varphi_n)_T^\circ + 2^{-k}]$ . Hence defining the effective procedure to output  $m$  on inputs  $n$  and  $k$  suffices.  $\square$

Let  $L^+$  be the Henkin extended signature effectively given by Lemma 3, and let  $(\theta_n)_{n \in \mathbb{N}}$  be an effective enumeration of the  $L^+$ -wffs. The next lemma we present is similar to Lemma 4.7 in Calvert [5]. However, in our case, the construction is with respect to any *name* of an  $L$ -theory,  $X \in \mathbb{N}^{\mathbb{N}}$ . Moreover, careful consideration is taken with respect to when two  $L^+$ -sentences are provably equivalent with respect to a given  $L^+$ -theory.

The basic idea is the following. Given the degree of truth of a theory  $\cdot_T^\circ$ , find the first  $L^+$ -wff of the form  $\varphi \dot{-} \psi$  which has a *strictly positive* degree of truth. It follows that

there is some structure  $\mathfrak{M}$  and assignment  $\sigma$  such that  $\mathfrak{M}, \sigma \models T$  while  $\mathfrak{M}, \sigma \not\models \varphi \dot{\div} \psi$ . Hence  $\mathfrak{M}, \sigma \models T \cup \{\psi \dot{\div} \varphi\}$ . Moreover,  $\cdot^{\circ}_{T \cup \{\psi \dot{\div} \varphi\}}$  is shown to be effective in  $\cdot^{\circ}_T$ . Thus we may effectively complete  $T$  as an  $L^+$ -theory.

**Lemma 4** *There is an effective procedure which given  $X$ , a name of an  $L$ -theory  $T$ , outputs  $\Phi(X) \subseteq \mathbb{N}$  such that  $T \cup \{\theta_n : n \in \Phi(X)\}$  is consistent, and for every pair of  $L^+$ -wffs  $\varphi$  and  $\psi$ , either  $\varphi$  and  $\psi$  are provably equivalent with respect to  $T \cup \{\theta_n : n \in \Phi(X)\}$ , or exactly one of  $\varphi \dot{\div} \psi$  or  $\psi \dot{\div} \varphi$  is in  $\{\theta_n : n \in \Phi(X)\}$ .*

**Proof** We proceed via effective recursion to construct a partial computable function which sends names of  $L$ -theories to sets of natural numbers. First define  $\Phi_0(X) := \emptyset$ , for every  $X \in \mathbb{N}^{\mathbb{N}}$ . As the recursive assumption, we suppose that at stage  $s$ , if  $X$  is a name of an  $L$ -theory  $T$ , then  $\Phi_s(X)$  is defined, finite, and  $T \cup \{\theta_n : n \in \Phi_s(X)\}$  is consistent. At stage  $s + 1$ , the following procedure attempts to construct  $\Phi_{s+1}(X)$ .

For every pair of  $L^+$ -wffs  $\varphi$  and  $\psi$ , define the real number

$$r_{\varphi, \psi, X, s+1} := \left( \sup_{\vec{x}, \vec{y}} \sup_{\vec{z}} \left( \left( (\psi \dot{\div} \varphi) \dot{\div} \left( \bigvee_{n \in \Phi_s(X)} \theta_n \right) \right) [\vec{z}/\vec{c}] \right) \right)^{\circ}_T$$

where  $\vec{x}$  and  $\vec{y}$  are the free variables appearing in  $\psi \dot{\div} \varphi$  and  $\bigvee_{n \in \Phi_s(X)} \theta_n$ , respectively,  $\vec{c}$  is the (possibly empty) tuple of constants from  $L^+$  and not in  $L$  appearing in  $(\psi \dot{\div} \varphi) \dot{\div} \left( \bigvee_{n \in \Phi_s(X)} \theta_n \right)$ , and  $\vec{z}$  is a  $|\vec{c}|$ -tuple of variable symbols distinct from  $\vec{x}$  and  $\vec{y}$ . Notably, these real numbers are computable in  $X$ , uniformly in  $\varphi$ ,  $\psi$ , and  $s$ . To see this, notice that each recursively defined  $\Phi_s(X)$  is finite, each free variable becomes bound by the quantifier, and every constant from  $L^+$  not in  $L$  is replaced by a variable and bound. Hence each formula checked above is actually an  $L$ -sentence, so  $X$  can compute a rational approximation of  $r_{\varphi, \psi, X, s+1}$  within  $2^{-(s+2)}$ . Call such a rational  $q_{\varphi, \psi, X, s+1}$ . Then search the pairs of  $L^+$ -wffs for the first pair  $\varphi$  and  $\psi$  such that  $\varphi \dot{\div} \psi \notin \{\theta_n : n \in \Phi_s(X)\}$  and  $q_{\varphi, \psi, X, s+1} \geq 2^{-(s+1)}$ . By Proposition 2, there are infinitely many  $L^+$ -wffs  $\psi$  such that  $r_{0, \psi, X, s+1} = 1$ , and hence such that  $q_{0, \psi, X, s+1} \geq 2^{-(s+1)}$ . Thus, when  $X$  is a name of an  $L$ -theory, the procedure must halt. When such a pair  $\varphi$  and  $\psi$  is found, search the effective enumeration of the  $L^+$ -wffs for the index  $m$  of  $\varphi \dot{\div} \psi$  and define  $\Phi_{s+1}(X) := \Phi_s(X) \cup \{m\}$ . Clearly, if  $\Phi_{s+1}(X)$  is defined, it is also finite, by construction. We now claim that this  $\Phi = \bigcup_{s \in \mathbb{N}} \Phi_s$  witnesses the lemma.

Fix a name of an  $L$ -theory  $X \in \mathbb{N}^{\mathbb{N}}$ . To see that  $T \cup \{\theta_n : n \in \Phi(X)\}$  is consistent, we show that each  $T \cup \{\theta_n : n \in \Phi_{s+1}(X)\}$  is consistent, for every  $s \in \mathbb{N}$ . We proceed inductively.

Suppose  $T \cup \{\theta_n : n \in \Phi_s(X)\}$  is consistent and fix  $m \in \Phi_{s+1}(X) \setminus \Phi_s(X)$ . By construction this  $m$  is the index for  $\varphi \dot{\div} \psi$  where  $q_{\varphi, \psi, X, s+1} \geq 2^{-(s+1)}$ . It follows by the definition of  $q_{\varphi, \psi, X, s+1}$  and Propositions 3 and 4, Corollary 2, logical equivalence, and the Deduction Theorem, that we have each of the following:

$$\begin{aligned} & \left( (\psi \dot{\div} \varphi) \dot{\div} \left( \bigvee_{n \in \Phi_s(X)} \theta_n \right) \right)_T^\circ \geq 2^{-(s+2)} \\ \implies & \left( 2^{-(s+2)} \dot{\div} \left( (\psi \dot{\div} \varphi) \dot{\div} \left( \bigvee_{n \in \Phi_s(X)} \theta_n \right) \right) \right)_T^\circ = 0 \\ \implies & T \vdash \left( 2^{-(s+2)} \dot{\div} \left( (\psi \dot{\div} \varphi) \dot{\div} \left( \bigvee_{n \in \Phi_s(X)} \theta_n \right) \right) \right) \dot{\div} 2^{-(s+3)} \\ \implies & T \cup \{\theta_n : n \in \Phi_s(X)\} \cup \{\psi \dot{\div} \varphi\} \vdash 2^{-(s+3)} \end{aligned}$$

Therefore,  $T \cup \{\theta_n : n \in \Phi_s(X)\} \cup \{\psi \dot{\div} \varphi\}$  is inconsistent. It follows by Lemma 1 that  $\varphi \dot{\div} \psi$  is consistent with  $T \cup \{\theta_n : n \in \Phi_s(X)\}$ .

Hence, we need only show that for every pair of  $L^+$ -wffs  $\varphi$  and  $\psi$ , either  $\varphi$  and  $\psi$  are provably equivalent with respect to  $T \cup \{\theta_n : n \in \Phi(X)\}$ , or exactly one of  $\varphi \dot{\div} \psi$  or  $\psi \dot{\div} \varphi$  is in  $\{\theta_n : n \in \Phi(X)\}$ .

Note that a pair of  $L^+$ -wffs  $\varphi$  and  $\psi$  is provably equivalent with respect to  $T \cup \{\theta_n : n \in \Phi(X)\}$  if and only if for every  $s \in \mathbb{N}$ , there is some  $S \in \mathbb{N}$  such that

$$T \cup \{\theta_n : n \in \Phi_S(X)\} \vdash (\varphi \dot{\div} \psi) \dot{\div} 2^{-(s+2)} \quad \text{and} \quad T \cup \{\theta_n : n \in \Phi_S(X)\} \vdash (\psi \dot{\div} \varphi) \dot{\div} 2^{-(s+2)}.$$

Now fix a pair of  $L^+$ -wffs  $\varphi$  and  $\psi$  that are not provably equivalent with respect to  $T \cup \{\theta_n : n \in \Phi(X)\}$ . Then there must be some  $s \in \mathbb{N}$  such that for every  $S \in \mathbb{N}$ , either

$$T \cup \{\theta_n : n \in \Phi_S(X)\} \not\vdash (\varphi \dot{\div} \psi) \dot{\div} 2^{-(s+2)} \quad \text{or} \quad T \cup \{\theta_n : n \in \Phi_S(X)\} \not\vdash (\psi \dot{\div} \varphi) \dot{\div} 2^{-(s+2)}.$$

At least one of these two cases must hold for infinitely many  $S \in \mathbb{N}$ . Without loss of generality, since the cases are symmetric, suppose it is the latter. It follows by Corollary 2 that for every  $S \in \mathbb{N}$ ,  $(\psi \dot{\div} \varphi)_{T \cup \{\theta_n : n \in \Phi_S(X)\}}^\circ \geq 2^{-(s+2)}$ . Hence by Proposition 1, for every  $S \in \mathbb{N}$ :

$$\left( (\psi \dot{\div} \varphi) \dot{\div} \left( \bigvee_{n \in \Phi_S(X)} \theta_n \right) \right)_T^\circ \geq 2^{-(s+2)}$$

Then by Propositions 3 and 4, for every  $S \in \mathbb{N}$ ,  $r_{\varphi, \psi, X, S+1} \geq 2^{-(s+2)}$ . Thus for some  $S \geq s + 2$ ,  $\varphi$  and  $\psi$  will have to be the first pair such that  $\varphi \dot{\div} \psi \notin \{\theta_n : n \in \Phi_S(X)\}$  and  $q_{\varphi, \psi, X, S+1} \geq 2^{-(s+3)} \geq 2^{-(S+1)}$ . It follows that the procedure will place the index for  $\varphi \dot{\div} \psi$  into  $\Phi_{S+1}(X)$ , so  $\varphi \dot{\div} \psi \in \{\theta_n : n \in \Phi(X)\}$ .  $\square$

It should be noted that if  $T$  is not complete, a name of  $T$  does not specify a *unique* consistent extension of  $T$ . The above procedure constructs a complete extension, which itself has a unique maximally consistent extension, but the procedure is dependent on the enumeration of the  $L^+$ -wffs. When that enumeration changes, if  $T$  is not a complete theory, the above extension of  $T$  may also change.

### 4.3 Generalized effective completeness

We now come to our main result.

**Theorem 7** (Generalized Effective Completeness) *There is an effective procedure which, given a name  $X \in \mathbb{N}^{\mathbb{N}}$  of an  $L$ -theory  $T$ , produces a presentation of an  $L^+$ -structure  $\mathfrak{M}$  such that  $\mathfrak{M} \models T$ .*

**Proof** Compute  $L^+$  as in Lemma 3. Given a name of an  $L$ -theory  $X \in \mathbb{N}^{\mathbb{N}}$ , let  $\Phi(X)$  be as in Lemma 4. Then, by Theorem 3, extend  $T \cup \{\theta_n : n \in \Phi(X)\}$  to a maximally consistent, Henkin complete  $L^+$ -theory  $\Gamma$ . By Proposition 5,  $\mathfrak{M}_\Gamma \models T$ .

Since  $L^+$  is effectively numbered, the set of constants of  $L^+$  is also effectively numbered, which we may effectively join to an effective numbering of the variable symbols. Let  $g'$  be such an effective numbering. Then, for every  $n \in \mathbb{N}$ , define  $g(n) := [g'(n)]$ , the equivalence class of  $g'(n)$  in  $|\mathfrak{M}_\Gamma|$ . By construction, the algebra generated by  $\text{ran}(g)$  in  $\mathfrak{M}_\Gamma$  is the set of all equivalence classes of terms of  $L^+$ , that is, equivalence classes of the elements of  $|\mathfrak{M}'_\Gamma|$ . It follows that this algebra is dense in  $|\mathfrak{M}_\Gamma|$ , since by construction  $|\mathfrak{M}_\Gamma|$  is the metric completion of  $|\mathfrak{M}'_\Gamma|$ . Thus  $(\mathfrak{M}_\Gamma, g)$  is a presentation of  $\mathfrak{M}_\Gamma$ . We further claim that  $(\mathfrak{M}_\Gamma, g)$  is a computable presentation.

To see this, fix a code of an arbitrary  $N$ -ary predicate symbol  $P$ , codes of rational points  $[t_0], \dots, [t_{N-1}]$ , and  $k \in \mathbb{N}$ . From these, use  $g'$  to decode  $L^+$ -terms  $t_0, \dots, t_{N-1}$  corresponding to  $[t_0], \dots, [t_{N-1}]$ . Then execute the following.

Compute the finite set  $D = \{p \in \text{Dyad} : \text{the denominator of } p \text{ is less than } 2^{k+2}\}$ . By the construction of  $\Phi(X)$ , with access to an oracle that computes  $X$  we may compute the least  $M \geq k + 2$  such that for all but one  $p \in D$ , exactly one of  $p \div P(t_0, \dots, t_{N-1})$  or  $P(t_0, \dots, t_{N-1}) \div p$  is in  $\{\theta_n : n \in \Phi_{M+1}(X)\}$ .<sup>4</sup> Then compute the finite set

<sup>4</sup>It may be that for some  $p \in D$ , for every  $M \in \mathbb{N}$ ,  $(P(t_0, \dots, t_{N-1}) \div \bigvee_{n \in \Phi_M(X)} \theta_n)_T^\circ$  and  $p$  differ by less than  $2^{-(M+2)}$ . This can only occur if  $P(t_0, \dots, t_{N-1})$  and  $p$  are provably equivalent with respect to  $T \cup \{\theta_n : n \in \Phi(X)\}$ , which can happen for at most one  $p \in D$ , since  $T \cup \{\theta_n : n \in \Phi(X)\}$  is consistent.

$E = \{p \in D : p \dot{\div} P(t_0, \dots, t_{N-1}) \in \{\theta_n : n \in \Phi_{M+1}(X)\}\}$ . Notice that by construction:

$$\max_{p \in E} p \dot{\div} P(t_0, \dots, t_{N-1}) \in \Gamma \text{ and } P(t_0, \dots, t_{N-1}) \dot{\div} \left( \min_{p \in D \setminus E} p \dot{+} 2^{-(k+2)} \right) \in \Gamma$$

Therefore:

$$\mathfrak{M}_\Gamma \models \max_{p \in E} p \dot{\div} P(t_0, \dots, t_{N-1}) \text{ and } \mathfrak{M}_\Gamma \models P(t_0, \dots, t_{N-1}) \dot{\div} \left( \min_{p \in D \setminus E} p \dot{+} 2^{-(k+2)} \right)$$

It follows that:

$$\max_{p \in E} p \leq (P(t_0, \dots, t_{N-1}))^{\mathfrak{M}_\Gamma} \leq \left( \min_{p \in D \setminus E} p + 2^{-(k+2)} \right)$$

This implies that:

$$(P(t_0, \dots, t_{N-1}))^{\mathfrak{M}_\Gamma} \in \left[ \left( \min_{p \in D \setminus E} p - 2^{-(k+1)} \right), \left( \min_{p \in D \setminus E} p + 2^{-(k+1)} \right) \right]$$

Note lastly that this procedure was uniform in  $X \in \mathbb{N}^{\mathbb{N}}$ . □

It follows that the presentation given by the above theorem is Turing reducible to the name input. Hence if the name given is computable (meaning the theory is decidable), the presentation produced is also computable. Standard effective completeness then comes as a corollary.

**Corollary 3** (Effective Completeness of Continuous Logic) *Every decidable theory is modeled by a computably presentable structure.*

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