Journal of Logic & Analysis 17:7 (2025) 1–40
ISSN 1759-9008

Representation of unlimited integers as the product of integers with some constraints

BELLAOUAR DJAMEL
BOUDAOUD ABDELMADJID

Abstract: We say that two unlimited natural numbers x and y have the same order if x/y is appreciable. As a continuation of our previous papers [4, 5], we study the representation of unlimited natural numbers as the sum of a limited integer and the product of at least two unlimited natural numbers ω_1, ω_2 that satisfy additional requirements, such that ω_1 and ω_2 have the same order, $\gcd(\omega_1, \omega_2) = 1$, or both.

2020 Mathematics Subject Classification 26E35,03H05 (primary); 11A51, 11A41, 11B83. (secondary)

Keywords: Representation of integers, Internal set theory, Greatest common divisor, Prime numbers, Factorization, appreciable rationales

1 Introduction

Several representations of natural numbers have been studied for a long time in Number Theory. For example, representing integers in different bases, representing integers as the sum of a fixed number of squares or more generally as the quadratic progressions, etc (see, for example, Johnson [24], Grosswald [18], and Nathanson [28]). Ramanujan [32] studied the number of representations of n as a sum of k squares or the number of representations of n as a sum of k triangular numbers.

Throughout this work, $2 = p_1 < p_2 < ... < p_n < ...$ will denote the successive prime numbers. The sequence $q_1 < q_2 < ... < q_n < ...$ denotes an arbitrary sequence of primes. The fundamental theorem of arithmetic states that any natural number that is greater than 1 can be factored into a product of prime numbers. That is, any natural number $n \geq 2$ can be represented as $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$, where $2 \leq q_1 < q_2 < \cdots < q_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k$ are positive integers. Also, there are several patterns representing each natural number, some of which give a representation of some families of integers in infinitely many ways. We mention the most famous of them.

- The representation of prime numbers by the binary quadratic forms $x^2 + y^2$ and $x^2 + 2y^2$. For more details, see Buchmann and Vollmer [9], Nathanson [28, Section 13.2, page 404]. On the other hand, suppose that $q \mid (n^2 + 2)$ for some $n \in \mathbb{N}$ relatively prime to q. By Mollin [27, page 46-47], there exist unique $x, y \in \mathbb{N}$ such that $q = x^2 + 2y^2$. In addition, if $q \mid (n^2 - 2)$ for some $n \in \mathbb{N}$ relatively prime to q, then actually one can prove that q can be represented in the form $x^2 - 2y^2$ in infinitely many ways.
- For every m, every even number can be represented in infinitely many ways as a difference of positive integers relatively prime with m (see, eg, Sierpinski [35, page 4, Problem 49]).
- In [2, page 163], Erdös and Suranyi proved that every integer n can be represented in infinitely many ways in the form $n = \pm 1^2 \pm 2^2 \pm \ldots \pm m^2$ for some positive integer m and some choice of signs $+$ or $-$.
- The m-adic representation of n (see Nathanson [28, Theorem 1.2, page 5]).
- In view of Andrica and Andreescu [3, page 395], every positive integer n has the unique factorial base expansion:

$$n = 1! \cdot f_1 + 2! \cdot f_2 + \ldots + m! \cdot f_m,$$

  where each f_i is an integer, $0 \leq f_i \leq i$, and $f_m > 0$.
- In Zeckendorf [37], it is proved that every positive integer is a sum of distinct terms of Fibonacci sequence.

Factoring a composite integer is believed to be a hard problem. Factoring a positive integer n means finding positive integers u and v such that the product of u and v equals n, and such that both u and v are greater than 1. Such u and v are called factors (or divisors) of n, and $n = u \cdot v$ is called a factorization of n. Recall that the problem of finding factors of a number $n = 2k + 1$ is solved if we can express n as $x^2 - y^2$ in a nontrivial way (by trivial we mean $2k + 1 = (k + 1)^2 - k^2$).

Leibniz, Euler and Cauchy were among the first to use infinitely small quantities. In order to make better use of this idea, In 1961 Robinson presented another approach, namely, *non-standard analysis*. In 1977, Nelson provided another presentation of non-standard analysis, called IST (Internal Set Theory), based on ZFC and to which a new unary predicate called *standard* was added. The use of this predicate is governed by the following three axioms: *Transfer principle*, *Idealization principle* and *Standardization principle*. This is the framework within which we present this work. For details, see Diener [13, pages 1-19], Diener and Reeb [14, pages 7–33], Nilson [29] and Robinson [34].

Now, assume that $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r}$ is an unlimited positive integer. We cannot generally deduce that n can be written in the form $\omega_1 \omega_2$, where ω_1, ω_2 are formed using some of the prime powers $q_1^{\alpha_1}, q_2^{\alpha_2}, ..., q_r^{\alpha_r}$ with ω_1/ω_2 appreciable, even when r is unlimited. So we cannot easily deduce from the fundamental theorem of arithmetic that n can be written as the sum of a limited integer and the product of at least two unlimited natural numbers having the same order. In addition, we cannot generally deduce from this theorem that any unlimited natural number n can be written as the sum of a nonzero limited integer and the product of pairwise relatively prime unlimited integers, especially for the numbers of the form $s \cdot q$, where $s \geq 1$ is limited and q is a prime number, and the numbers of the form $p_1 p_2 ... p_n$, where n is unlimited and p_i is the i-th prime number. This raises the following natural question: Can every unlimited natural number n be represented in the form $n = s + \omega_1\omega_2$, where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited and satisfy at least one of the following conditions: (i) $\gcd(\omega_1, \omega_2) = 1$ and (ii) ω_1/ω_2 is appreciable?

Our purpose in this paper is to characterize some families of unlimited natural numbers which can be written in the form $n = s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\gcd(\omega_1, \omega_2) = 1$ or ω_1/ω_2 appreciable. We present further some examples that satisfy the two conditions. Thus, we need the following definition:

Definition 1.1 Two real numbers x and y are of the same order, written $x \sim y$, if x/y is appreciable. Or, equivalently, there exist standard real numbers $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 < |x/y| < r_2$.

In our paper [5], we considered the following representation form:

$$(\mathrm{F}_2) \qquad n = s + \omega_1\omega_2,$$

where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited. In fact, we proved that there are several types of integers satisfying (F_2); but unfortunately, we could not give a representation of every unlimited positive integer as in (F_2). In the same context, a great contribution to this representation was made by Hrbáček [21], who proved a remarkable result: Assuming Dickson's Conjecture, there are unlimited primes that fail to satisfy (F_2).

Throughout this paper, we look at a similar representation by adding some conditions, such as $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In other words, the main goal of the present paper is to characterize some families of unlimited natural numbers which can be written as $n = s + \omega_1\omega_2$, where $s \in \mathbb{Z}$ is limited and ω_1, ω_2 satisfy one of the above conditions. Symbolically: Let us consider the following form:

$$(\mathrm{A}_2) \qquad \begin{cases} n = s + \omega_1\omega_2 \\ \omega_1 \sim \omega_2 \end{cases}$$

where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with ω_1/ω_2 appreciable.

We give two typical examples:

Example 1.2 Let ω be unlimited. We see that

$$1 + 10^{\omega} = \begin{cases} 2 + \left(10^{\frac{\omega}{2}} - 1\right)\left(10^{\frac{\omega}{2}} + 1\right) \text{ if } \omega \text{ is even,} \\ 11 + 10\left(10^{\frac{\omega-1}{2}} - 1\right)\left(10^{\frac{\omega-1}{2}} + 1\right), \text{ otherwise.} \end{cases} \tag{1}$$

Example 1.3 Let $l \geq 11$ be a limited positive integer and let ω be an unlimited positive integer. It is clear that $n = 10^{\omega} - l$ is of the form $s + \omega_1\omega_2$, where $s \in \mathbb{Z}_-^*$ is limited and ω_1, ω_2 are two unlimited positive integers with $\omega_1 \sim \omega_2$. Then we can prove that n in the form $s + \omega_1\omega_2$, where $s \in \mathbb{Z}_+^*$ is limited and ω_1, ω_2 are two unlimited positive integers with ω_1/ω_2 appreciable. In fact, fix a limited k so that $l < 10^k$. We see that $n = 10^k(10^{\omega-k} - 1) + 10^k - l$, and the result follows by the explicit factoring of the latter similarly as in (1).

Similar to the work of Hrbáček [21], by assuming Dickson's Conjecture, we will prove in Theorem 3.2 that there are infinitely many unlimited integers of the form (F_2) which fail to satisfy (A_2). If ω_1, ω_2 are relatively prime, then (F_2) becomes:

$$\begin{cases} n = s + \omega_1\omega_2 \\ \gcd(\omega_1, \omega_2) = 1. \end{cases} \tag{R_2}$$

If ω_1, ω_2 are both relatively prime and have the same order, then (F_2) becomes:

$$\begin{cases} n = s + \omega_1\omega_2 \\ \omega_1 \sim \omega_2 \\ \gcd(\omega_1, \omega_2) = 1. \end{cases} \tag{AR_2}$$

Example 1.4 Let $a = 2^x \cdot 5^y$, where $x, y \geq 0$ are limited, and let ω be unlimited. Then $a\sum_{i=0}^{\omega} 10^{\omega-i}$ is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{N}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. We see that

$$a\sum_{i=0}^{\omega} 10^{\omega-i} = a\left(\frac{10^{\omega+1} - 1}{9}\right) = \begin{cases} a + \dfrac{10a(10^{\frac{\omega}{2}} - 1)}{9}(10^{\frac{\omega}{2}} + 1) \text{ if } \omega \text{ is even,} \\ \dfrac{a(10^{\frac{\omega+1}{2}} - 1)}{9}(10^{\frac{\omega+1}{2}} + 1), \text{ otherwise.} \end{cases}$$

When ω is even we let $s = a$, $\omega_1 = 10a(10^{\frac{\omega}{2}} - 1)/9$ and $\omega_2 = 10^{\frac{\omega}{2}} + 1$. When ω is odd we let $s = 0$, $\omega_1 = a(10^{\frac{\omega+1}{2}} - 1)/9$ and $\omega_2 = 10^{\frac{\omega+1}{2}} + 1$. Clearly, $\gcd(\omega_1, \omega_2) = 1$ since $2, 5 \nmid (10^t + 1)$ for $t \geq 1$.

In the rest of this paper the letter k always stands for a limited integer > 0. Now, for every limited $k \geq 2$ we consider the general form of (A$_2$):

$$(\mathrm{A}_k) \qquad \begin{cases} n = s + \omega_1\omega_2...\omega_k \\ \omega_1 \sim \omega_2 \sim ... \sim \omega_k \,, k \geq 2 \end{cases}$$

where $s \in \mathbb{Z}$ is limited and the positive integers $\omega_1, \omega_2, ..., \omega_k \in \mathbb{N}$ are unlimited with ω_i/ω_j appreciable for $1 \leq i,j \leq k$. This implies that $\omega_i \sim \sqrt[k]{n}$ for $i = 1, 2, ..., k$. If in addition these positive integers are pairwise relatively prime[1], then (A$_k$) becomes:

$$(\mathrm{AR}_k) \qquad \begin{cases} n = s + \omega_1\omega_2...\omega_k \\ \omega_1 \sim \omega_2 \sim ... \sim \omega_k \\ \gcd(\omega_i, \omega_j) = 1, \text{ for } i \neq j. \end{cases}$$

So in this paper we are interested in unlimited positive integers that can be represented as in (A$_2$), (A$_k$) and (AR$_k$). Several types of integers satisfy one of the above forms, such as integer-valued polynomials whose leading coefficients are positive, Femat numbers, Merssene numbers, Cullen numbers and special numbers defined by recursive patterns such as Fibonacci numbers and their companion Lucas numbers.

Note that if q is an unlimited prime number we will prove that there exists a positive integer λ for which $\lambda \cdot q$ is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\gcd(\omega_1, \omega_2) = 1$ and $\omega_1 \sim \omega_2$ (see Proposition 8.3 with $\lambda = \omega = q$ and $b = 0$). A natural question, which remains open: Does there exist an unlimited prime number of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$. The answer is positive by assuming one of the conjectures: Bouniakowsky conjecture [8] or the $n^2 + 1$ conjecture [36] and many others. Moreover, we show in Section 11 that the answer to this question cannot be deduced from Dirichlet's Theorem on primes in arithmetic progressions (see Nathanson [28, page 347]). Using this theorem, we will prove (see Proposition 11.1) that there are unlimited prime numbers of the form (R$_2$).

For further research, we close this paper with a list of open questions which have arisen during our study.

[1] *A representation of an unlimited positive integer n of the form $n = s + \omega_1\omega_2...\omega_k$ is said to be "primitive" if $\omega_1, ..., \omega_k \in \mathbb{N}$ are unlimited with $\gcd(\omega_i, \omega_j,) = 1$ for $i \neq j$. This is similar to the representation stated in Mollin [27, Definition 6.1, page 247].*

2 Representation of unlimited integers using the division algorithm

In Nathanson [28, page 404], it is shown that for every positive integer n the diophantine equation $n = \omega_1\omega_2 + \omega_3^2$ has infinitely many solutions in integers $\omega_1, \omega_2, \omega_3$. In the following theorem we get a similar representation of any unlimited positive integer.

Theorem 2.1 *Every unlimited positive integer n can be represented as*

$$n = \omega_1 + \omega_2 + \omega_3\omega_4,$$

where ω_1, ω_2, ω_3 and ω_4 are unlimited positive integers with $\omega_i \sim \omega_j$ for $i \neq j$.

Proof Let $a = \left[\sqrt{n}\right]$ which is unlimited. Then by the division algorithm, there exist unique integers q and r such that $n = a \cdot q + r$ with $0 \leq r < a$. In this case, we see that $a \sim q$. In fact, since $a \sim \sqrt{n}$ and $(\sqrt{n}/a)^2 = q/a + r/a^2$, we conclude that q/a is appreciable. Therefore, we have:

$$n = a \cdot q + r = (a - 2 + 2)q + r = q + (q + r) + (a - 2)q.$$

We put $\omega_1 = q$, $\omega_2 = q + r$, $\omega_3 = a - 2$ and $\omega_4 = q$. Then $\omega_i \sim \omega_j$ for $i \neq j$, and the proof is finished. □

Remark 2.2 We deduce from Theorem 2.1 that every unlimited positive integer n can be represented as $n = s + \omega_1 + \omega_2 + \omega_3\omega_4$ where $s \in \mathbb{Z}^*$ is limited and $\omega_i \in \mathbb{N}$ are unlimited with $\omega_i \sim \omega_j$ for $i \neq j$.

Proposition 2.3 *Given any unlimited positive integer n, there exists an integer-valued polynomial $p(x)$ of degree 2 with limited leading coefficient and an unlimited positive integer m such that $n = p(m)$. Moreover, n can be represented in one of the following two forms:*

I. $n = \omega_1 + \omega_2\omega_3$ *where* $\omega_1, \omega_2, \omega_3 \in \mathbb{N}$ *are unlimited with* $\omega_1 < \omega_2 \leq \omega_3$ *and* $\omega_2 \sim \omega_3$.

II. $n = s + \omega_1\omega_2$ *where* $s \in \mathbb{N}$ *is limited and* $\omega_1, \omega_2 \in \mathbb{N}$ *are unlimited with* $\omega_1 \sim \omega_2$.

Proof Let $m = \left[\sqrt{n}\right]$, where $[x]$ denotes the integer part of x. By writing n in basis m, we obtain

$$n = a_0 + a_1 m + a_2 m^2 + ... + a_k m^k, \tag{2}$$

where k is the nonnegative integer such that $m^k \leq n < m^{k+1}$ and $a_0, a_1, ..., a_k$ are integers such that $1 \leq a_k \leq m-1$ and $0 \leq a_i \leq m-1$ for $i = 0, 1, ..., k-1$. Since $m = \left[\sqrt{n}\right]$ is unlimited and $\sqrt{n} - 1 < m \leq \sqrt{n}$, we conclude that $m^2 \leq n < m^3$. Therefore, $k = 2$. It follows from (2) that $n = a_0 + m(a_1 + a_2 m)$ where a_2 is limited; otherwise,

$$1 = \frac{a_0}{n} + \frac{a_1 m}{n} + a_2 \left(\frac{m}{\sqrt{n}} \cdot \frac{m}{\sqrt{n}} \right) \cong \infty$$

which is a contradiction. Now, we distinguish two cases:

Case 1. Assume that a_0 is unlimited. We put $\omega_1 = a_0$, $\omega_2 = m$ and $\omega_3 = a_1 + a_2 m$. Since $0 \leq a_i \leq m-1$ for $i = 0, 1$ and $a_2 \in \mathbb{N}^*$ is limited, we have $\omega_1, \omega_2, \omega_3 \in \mathbb{N}$ are unlimited with $\omega_1 < \omega_2 \leq \omega_3$ and $\omega_2 \sim \omega_3$. Then n is in form I.

Case 2. Assum e that a_0 is limited. Here we put $s = a_0$, $\omega_1 = m$ and $\omega_2 = a_1 + a_2 m$. Therefore, $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$. Then n is in form II.

This completes the proof. □

3 On the numbers of the form (F_2) which are not of the form (A_2)

In this section, we will use the following Dickson's Conjecture (see Dickson [11]) to prove that there are infinitely many unlimited numbers of the form (F_2), but not of the form (A_2). In fact, this result is similar to the main theorem of Hrbáček [21] and Boudaoud [6].

By generalizing Dirichlet's theorem (see Nathanson [28, Theorem 10.9, page 347]) and concerning the simultaneous values of several linear polynomials, Dickson [11] stated the following conjecture in 1904:

Conjecture 3.1 (Dickson's conjecture) *Let $k \geq 1$ and $f_i(x) = a_i + b_i \cdot x$ with a_i and b_i integers, $b_i \geq 1$ (for $i = 1, ..., k$). Assume that there exists no integer > 1 dividing the products $f_1(n) f_2(n) ... f_k(n)$ for all positive integers n. Then there exist infinitely many positive integers m such that all the numbers $f_1(m), f_2(m), ..., f_s(m)$ are primes.*

Recall that Dickson's conjecture implies many important results (see Ribenboim [33]) such as: there exist infinitely many Sophie Germain primes or safe primes[2] from which

[2]Recall that a prime number p is a *Sophie Germain prime* if $2p+1$ is also prime (see Dubner [12]). The number $2p+1$ associated with a Sophie Germain prime is called a *safe prime*.

we can say that there are infinitely many primes of the form $1+2q$, where $q \cong +\infty$ is prime.

Theorem 3.2 *Assuming Dickson's conjecture, there exist infinitely many unlimited integers of the form* (F_2)*, none of which can be written as in* (A_2).

For the proof we need the following facts: For any positive integer n, there are at least n consecutive composite integers. Also by Bertrand's theorem we have $p_n < 2^n$ for all $n \geq 2$ where p_n is the n-th prime number.

Proof of Theorem 3.2 We use the fact that there exists an unlimited prime number p_γ such that $p_{\gamma+1} - p_\gamma \cong +\infty$, where $\gamma \in \mathbb{N}$ is also unlimited. Let ω be an unlimited positive integer such that ω/γ is unlimited. Let us construct the following system of polynomials:

$$\text{(3)} \qquad \begin{cases} P_{\alpha_1\alpha_2...\alpha_{\gamma-1}}(x) = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{\gamma-1}^{\alpha_{\gamma-1}} p_\gamma p_{\gamma+1} ... p_\omega \cdot x + 1 \\ P_{p_\gamma}(x) = p_1^\gamma p_2^\gamma ... p_{\gamma-1}^\gamma p_{\gamma+1} ... p_\omega \cdot x + p_\gamma \end{cases}$$

where $(\alpha_1, \alpha_2, ..., \alpha_{\gamma-1})$ runs the set $L = \{0, 1, ..., \gamma\}^{\gamma-1}$. So the system (3) is formed by $(\gamma+1)^{\gamma-1} + 1$ polynomials.

Now, we prove that the polynomials (3) satisfy the assumptions of Dickson's conjecture. First, note that $P_{\alpha_1\alpha_2...\alpha_{\gamma-1}}(x)$ and $P_{p_\gamma}(x)$ are integer-valued polynomials of degree 1 whose leading coefficients are positive. Assume further that there exists an integer $\widetilde{n} > 1$ dividing the product

$$\text{(4)} \qquad F(k) = \prod_{(\alpha_1,\alpha_2,...,\alpha_{\gamma-1}) \in L} P_{\alpha_1\alpha_2...\alpha_{\gamma-1}}(k) \cdot P_{p_\gamma}(k)$$

for all integer k. In particular, $\widetilde{n}$ divides the product (4) for $k = \widetilde{n}$. Since

$$\widetilde{n} \nmid P_{\alpha_1\alpha_2...\alpha_{\gamma-1}}(\widetilde{n})$$

holds for every $(\alpha_1, \alpha_2, ..., \alpha_{\gamma-1}) \in L$, we conclude that $\widetilde{n}$ divides $P_{p_\gamma}(\widetilde{n})$. Hence, $\widetilde{n}$ must be equal to p_γ. However, $\widetilde{n} = p_\gamma$ does not divide the product (4) for $k = 1$. By applying Dickson's conjecture, there exist infinitely many natural numbers $(m_i)_{i\geq 1}$ such that the numbers: $P_{\alpha_1\alpha_2...\alpha_{\gamma-1}}(m_i)$ and $P_{p_\gamma}(m_i)$ where $(\alpha_1, \alpha_2, ..., \alpha_{\gamma-1}) \in L$ are primes for all $i \geq 1$. By construction, assume that

$$N = P_{p_\gamma}(m_{i_0}) = p_1^\gamma p_2^\gamma ... p_{\gamma-1}^\gamma p_{\gamma+1} ... p_\omega m_{i_0} + p_\gamma$$

is prime for some positive integer i_0. Next we prove that for any standard integer $s \in \mathbb{Z}^*$, the number $N+s$ cannot be of the form (A$_2$), although it is written as the product of two unlimited integers, ie N is of the form (F$_2$). Indeed, we see that

$$N+s = p_1^{\gamma} p_2^{\gamma} ... p_{\gamma-1}^{\gamma} p_{\gamma+1} ... p_{\omega} m_{i_0} + p_{\gamma} + s = p_1^{\gamma} p_2^{\gamma} ... p_{\gamma-1}^{\gamma} p_{\gamma+1} ... p_{\omega} m_{i_0} + (p_{\gamma} + s).$$

If the prime power decomposition of $p_{\gamma}+s$ is given by $p_{\gamma}+s = p_1^{\ell_1} p_2^{\ell_2} ... p_{\gamma-1}^{\ell_{\gamma-1}}$ (note that in this product some exponents may be 0), then each ℓ_k $(1 \leq k \leq \gamma-1)$ is less than or equal to γ. In fact, using the construction above, $p_{\gamma}+s < p_{\gamma+1}$ and hence $p_{\gamma} < 2^{\gamma}$. So if $\ell_k \geq \gamma$ for $1 \leq k \leq \gamma-1$, then $p_{\gamma}+s \geq 2^{\ell_k} \geq 2^{\gamma}$. This is a contradiction. Therefore,

$$\begin{aligned} N+s &= p_1^{\gamma} p_2^{\gamma} ... p_{\gamma-1}^{\gamma} p_{\gamma+1} ... p_{\omega} m_{i_0} + (p_{\gamma}+s) \\ &= (p_{\gamma}+s) \left[\frac{p_1^{\gamma} p_2^{\gamma} ... p_{\gamma-1}^{\gamma} p_{\gamma+1} ... p_{\omega} m_{i_0}}{(p_{\gamma}+s)} + 1 \right] \\ &= (p_{\gamma}+s)(p_1^{\gamma-\ell_1} p_2^{\gamma-\ell_2} ... p_{\gamma-1}^{\gamma-\ell_{\gamma-1}} p_{\gamma+1} ... p_{\omega} m_{i_0} + 1) = \omega_1 \omega_2, \end{aligned}$$

where $\omega_1 = p_{\gamma}+s$ and $\omega_2 = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{\gamma-1}^{\alpha_{\gamma-1}} p_{\gamma+1} ... p_{\omega} m_{i_0} + 1$. Clearly, ω_1 is unlimited and also the prime number ω_2 is unlimited prime by the above construction. Thus, we obtain

$$\frac{\omega_1}{\omega_2} = \frac{p_{\gamma}+s}{p_1^{\alpha_1} p_2^{\alpha_2} ... p_{\gamma-1}^{\alpha_{\gamma-1}} p_{\gamma+1} ... p_{\omega} m_{i_0} + 1} \cong 0, \tag{5}$$

ie ω_1/ω_2 is not appreciable. Finally, since ω_2 is the largest prime factor of $N+s$ we conclude from (5) that there is no other factorization of $N+s$ where ω_1/ω_2 is appreciable. This completes the proof of Theorem 3.2. □

Example 3.3 In [21], Hrbáček proved by assuming Dickson's conjecture that there exists an unlimited prime number p such that for every $s \in \mathbb{Z}$ limited, $p+s$ is of the form $s' \cdot \pi$ where $s' \in \mathbb{N}$ is limited and π is unlimited prime.

4 Representation of unlimited natural numbers as in (R$_2$)

The goal of this section is to solve the following problem stated in our earlier paper [5, Question 5].

Problem 4.1 *Let n be an unlimited positive integer of the form $s+\omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and ω_1, ω_2 are two unlimited positive integers. We ask if n is also of the form $s' + \omega_1' \omega_2'$ where $s' \in \mathbb{Z}^*$ is limited and $\omega_1', \omega_2' \in \mathbb{N}$ are unlimited and relatively prime.*

Theorem 4.2 *Let p be an unlimited prime number and let $\alpha \geq 3$ be an odd integer. If $n = s + \ell \cdot p^{\alpha}$, where $s \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ are limited, then n can be written in the form $n = s' + \omega_1\omega_2$ where $s' \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with* $\gcd(\omega_1, \omega_2) = 1$.

For the proof, we will need the following lemmas.

Lemma 4.3 *Let $R \geq 1$ be an integer. For every $k \geq 2$ we have*

$$R^k - 1 = (R-1)\left(\sum_{i=1}^{k-1}(R^{k-i}-1) + k\right). \tag{6}$$

For every $k \geq 2$ we also have

$$R^{2k+1} + 1 = (R+1)\left((R+1)\sum_{i=1}^{2k}(-1)^{i-1} iR^{2k-i} + 2k + 1\right). \tag{7}$$

Proof First, we show (6). Clearly, we see that

$$R^k - 1 = (R-1)\left(\sum_{i=1}^{k-1} R^{k-i} + 1\right) = (R-1)\left(\sum_{i=1}^{k-1}\left(R^{k-i}-1\right) + k\right).$$

Next, for the proof of (7) we use the principle of mathematical induction on k. For $k = 1$ we have $R^3 + 1 = (R+1)(R(R+1) - 2(R+1) + 3)$. Hence, the assertion is true for $k = 1$. Let $k \geq 2$ and assume that (7) is true for k. We shall prove it for $k+1$. Indeed, we see that

$$R^{2k+3} + 1 = R^2\left\{(R+1)\left(\sum_{i=1}^{k-1}\left(R^{k-i}-1\right) + 2k + 1\right) - 1\right\} + 1$$

from which it follows that

$$R^{2k+3} + 1 = (R+1)\left((R+1)\sum_{i=1}^{2k}(-1)^{i-1} iR^{2k+2-i} + R^2(2k+1)\right) - R^2 + 1. \tag{8}$$

Now, we add and subtract R from the right-hand side of (8) to obtain

$$\begin{aligned} R^{2k+3} + 1 \quad &= \quad (R+1)\left((R+1)\sum_{i=1}^{2k}(-1)^{i-1} iR^{2k+2-i} + R^2(2k+1)\right) \\ &\quad -R(R+1) + (R+1) \end{aligned}$$

and hence

$$R^{2k+3} + 1 = (R+1)\left((R+1)\sum_{i=1}^{2k}(-1)^{i-1} iR^{2k+2-i} + R^2(2k+1) - R + 1\right). \tag{9}$$

By adding the quantity $R(2k+1) - R(2k+1) + (2k+1) - (2k+1) + 1 - 1 = 0$ to the second factor in the brackets of (9), we get

$$\begin{aligned} R^{2k+3}+1 &= (R+1)\left((R+1)\sum_{i=1}^{2k}(-1)^{i-1}\,iR^{2k+2-i} + R^2(2k+1) + R(2k+1)\right) \\ &\quad + (R+1)(-R(2k+1) + (2k+1) - (2k+1) + 1 - 1 - R + 1) \\ &= (R+1)\left((R+1)\sum_{i=1}^{2k}(-1)^{i-1}\,iR^{2k+2-i} + R^2(2k+1) + (2k+1)R\right) \\ &\quad + (R+1)(-(2k+2)(R+1) + 2(k+1) + 1) \\ &= (R+1)\left((R+1)\sum_{i=1}^{2k+2}(-1)^{i-1}\,iR^{2k+2-i} + 2k+3\right). \end{aligned}$$

This proves (7). The proof of Lemma 4.3 is finished. □

Lemma 4.4 *If ℓ is limited and λ_1, λ_2 are relatively prime, then $\ell \cdot \lambda_1\lambda_2 = \omega_1\omega_2$ where ω_1, ω_2 are relatively prime.*

Proof Let $\ell = q_1^{\beta_1}...q_r^{\beta_r}$. Then let $\omega_1 = q_{i_1}^{\beta_{i_1}}...q_{i_s}^{\beta_{i_s}} \cdot \lambda_1$, where $q_{i_1}, ..., q_{i_s}$ are those primes among $q_1, ..., q_r$ that do not divide λ_2, and let $\omega_2 = q_{j_1}^{\beta_{j_1}}...q_{j_{r-s}}^{\beta_{j_{r-s}}} \cdot \lambda_2$ where $q_{j_1}, ..., q_{j_{r-s}}$ are those primes among $q_1, ..., q_r$ that do not divide λ_1. The proof is finished. □

Proof of Theorem 4.2 Let $q \geq 3$ be a prime divisor of α. Note the existence of such prime divisor because $\alpha \geq 3$ is odd. We let $q = 2s+1$, where $s \geq 1$. Since we are going to apply the previous lemma, we put $R = p^{\frac{\alpha}{q}}$. Then $p^\alpha = R^q$. Applying (7), we have

$$R^q + 1 = R^{2s+1} + 1 = (R+1)\left((R+1)\sum_{i=1}^{2s}(-1)^{i-1} iR^{2s-i} + q\right). \tag{10}$$

Let $\omega_1 = R+1$ and let ω_2 be the second parenthesis in equation (10); clearly they are unlimited. We have $p^\alpha + 1 = \omega_1\omega_2$ and the only prime that can divide both ω_1 and ω_2 is q. So if q does not divide ω_1, then ω_1 and ω_2 are relatively prime and we are done. If q divides $\omega_1 = R+1$, then it does not divide $R-1$ because $q \geq 3$. By (6), we have

$$R^q - 1 = (R-1)\left(\sum_{i=1}^{q-1}(R^{q-i} - 1) + q\right). \tag{11}$$

Then let $\omega_1' = R - 1$ and ω_2' be the second parenthesis in (11). Then ω_1' and ω_2' are relatively prime because obviously no prime other that q could divide them both, and q does not divide ω_1'. Since $p^\alpha = \pm 1 + \omega_1\omega_2$. So $s + \ell \cdot p^\alpha = (s \pm \ell) + \ell\omega_1\omega_2$, and Theorem 4.2 now follows from Lemma 4.4. □

Now we are in a position to give the main result of this fourth section:

Theorem 4.5 *Let n be an unlimited positive integer of the form (F_2). Then n can be written in the form (R_2).*

Proof Let n be an unlimited positive integer of the form (F_2), ie $n = s + \omega_1\omega_2$, where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited. The canonical prime factorization of $\omega_1\omega_2$ allows us to write $\omega_1\omega_2 = q_1^{\alpha_1} q_2^{\alpha_2} ... q_t^{\alpha_t}$, where $q_1 < q_2 < ... < q_t$ and $\alpha_1, \alpha_2, ..., \alpha_t$ are positive integers. We distinguish the following cases:

Case 1. $t \cong +\infty$. There exist $i < j$ such that $q_i^{\alpha_i}$ and $q_j^{\alpha_j}$ are both unlimited. Here, we let $\omega_1' = q_j^{\alpha_j}$ and $\omega_2' = \omega_1\omega_2 / q_j^{\alpha_j}$.

Case 2. There is a unique i such that $q_i^{\alpha_i}$ is unlimited. Then t is limited and we can write $\omega_1\omega_2 = \ell \cdot q^\alpha$ where $q = q_i$ and $\alpha = \alpha_i$. We distinguish the following subcases:

Subcase A. α is even. Then we write $\alpha = 2\beta$ and $\omega_1\omega_2 = l + l \cdot (q^\beta + 1)(q^\beta - 1)$. If $\lambda_1 = q^\beta + 1$ and $\lambda_2 = q^\beta - 1$ are relatively prime we apply Lemma 4.4. Otherwise, they have a common factor 2 and we write

$$\omega_1\omega_2 = \ell + 4\ell \cdot \left(\frac{q^\beta + 1}{2}\right)\left(\frac{q^\beta - 1}{2}\right)$$

and we apply Lemma 4.4.

Subcase B. α is odd. There are two possibilities:

B.1. q is unlimited. This is Theorem 4.2.

B.2. q is limited, hence α is unlimited. Then we write $\alpha = 2\beta + 1$ and

$$\omega_1\omega_2 = q\ell + q\ell \cdot (q^\beta + 1)(q^\beta - 1).$$

We use the same argument as in *Subcase A* (with $q\ell$ in place of ℓ). This completes the proof. □

The following corollary shows that every unlimited prime power can expressed as in (R_2).

Corollary 4.6 *Let $n = q^\alpha$ be an unlimited prime power with $\alpha \geq 2$. Then n can be written in the form $n = s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\gcd(\omega_1, \omega_2) = 1$. That is, n can be represented as in (R_2).*

Proof Since $n = q^\alpha$ can be written as in (F_2), it follows from Theorem 4.5 that n can also be interpreted in the form (R_2). □

Note that the above Corollary fails for $\alpha = 1$ (assuming Dickson's conjecture).

5 On factoring of unlimited terms of some linear recurrence sequences

In this section we provide some unlimited terms of recurrence sequences which can be written as the sum of a limited integer and the product of two unlimited positive integers having the same order. Results along these lines can be found in Boudaoud [6, 7].

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence defined as $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, where $F_1 = F_2 = 1$. Its companion Lucas sequence $(L_n)_{n\geq 1}$ follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_1 = 1$ and $L_2 = 3$. The Pell sequence $(P_n)_{n\geq 1}$ is the binary recurrent sequence given by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 3$, where $P_1 = 1$ and $P_2 = 2$. Its companion Pell-Lucas sequence $(Q_n)_{n\geq 1}$ follows the same recursive pattern as the Pell numbers, but with initial values $Q_1 = 1$ and $Q_2 = 3$. Also, the generalized Fibonacci sequence $(G_n)_{n\geq 1}$ is given by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 3$, where $G_1 = a$ and $G_2 = b$ (here a and b are two limited positive integers). For details, see Guy [19, page 18], Koshy [25, pages 18,23] and Koshy [26, page 109].

The representation of unlimited numbers as in (F_2) was introduced by Boudaoud [6], which was the first beginning to give examples of numbers written as in (F_2). By applying the same argument as in [7], we can prove that for unlimited n, each of the integers F_n and L_n is of the form (F_2). But the question we ask here: We ask whether for unlimited n, each of the integers F_n, L_n, P_n, Q_n and G_n is of the form (A_2).

We prove the following theorem.

Theorem 5.1 *Let ω be unlimited. Each of the integers $F_\omega, L_\omega, P_\omega, Q_\omega$ and G_ω can be written in the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$.*

Proof **1)** Concerning F_ω. We distinguish two cases:

a) ω is odd. We put $m = (\omega - 1)/2 + 1$ and $n = (\omega - 1)/2$. By [26, page 97, identity 39] we have

$$5F_n^2 + 4(-1)^n = L_n^2, \tag{12}$$

from which it follows that L_n/F_n is appreciable (note that (12) holds for every $n \geq 1$). Moreover, since $F_\omega = F_{m+n}$, by Koshy [26, page 97, identity 56] we also have

$$F_\omega = F_{m+n} = \begin{cases} -F_{m-n} + L_n F_m, & \text{if } n \text{ is even} \\ -F_{m-n} + L_m F_n, & \text{otherwise.} \end{cases} \tag{13}$$

We will now consider separately the two subcases n is even and n is odd:

a-1) n is even. In view of (13), we show that L_n/F_m is appreciable. Indeed, we have $F_m = F_{m-1} + F_{m-2}$, and so $F_m/F_n = F_m/F_{m-1} = 1 + F_{m-2}/F_{m-1}$. Hence $1 < F_m/F_n < 2$, ie F_n/F_m is appreciable. Since

$$\frac{L_n}{F_m} = \frac{L_n}{F_n} \cdot \frac{F_n}{F_m},$$

we have L_n/F_m is also appreciable. Thus, since $F_{m-n} = F_1 = 1$, we have

$$F_\omega = -1 + L_n F_m$$

with $L_n \sim F_m$.

a-2) n is odd. Similarly, by applying (12), L_m/F_m is appreciable and since

$$L_m/F_n = L_m/F_m \cdot F_m/F_n,$$

we get L_m/F_n is appreciable. Thus, $F_\omega = -1 + L_m F_n$ with $L_m \sim F_n$.

b) ω is even. Put $m = (\omega - 2)/2 + 2$, $n = (\omega - 2)/2$. Then $\omega = m + n$. By [26, page 97, identity 39], we also have (13). We distinguish two subcases:

b-1) n is odd. Applying (12), L_n/F_n is appreciable. On the other hand, $m - 1 = n + 1$ and $m - 2 = n$, we have $L_m = L_{m-1} + L_{m-2} = L_{n+1} + L_n = 2L_n + L_{n-1}$. Hence, $L_m/L_n = 2 + L_{n-1}/L_n$, ie $2 < L_m/L_n < 3$. Then L_m/L_n is also appreciable. Thus, $L_m/F_n = L_m/L_n \cdot L_n/F_n$ is appreciable. In this case, $F_\omega = -1 + L_m F_n$, where $L_m \sim F_n$.

b-2) n is even. As in above, from (12), L_m/F_m is appreciable. Also from above L_m/L_n is appreciable. Hence, L_n/F_m is also appreciable, and so $F_\omega = -1 + L_n F_m$ where $L_n \sim F_m$.

2) Concerning L_ω. We distinguish two cases:

a) ω is odd. We put $m = (\omega+1)/2 + 1$, $n = (\omega+1)/2$. Then $L_\omega = L_{m+n}$. By [26, page 91, identity 86]

$$L_\omega = L_{m+n} = \begin{cases} L_{m-n} + L_m L_n, & \text{if } n \text{ is odd} \\ L_{m-n} + 5F_m F_n, & \text{otherwise.} \end{cases} \tag{14}$$

There are two possibilities:

a-1) n is odd. Using (14), $L_\omega = 1 + L_m L_{m-1}$, where L_m and L_{m-1} are of the same order since $L_m = L_{m-1} + L_{m-2} = L_{m-1}(1 + L_{m-2}/L_{m-1})$.

a-2) n is even. By (14), $L_\omega = 1 + 5F_m F_{m-1}$.

b) ω is even. Put $m = (\omega+2)/2 + 2$, $n = (\omega-2)/2$. Then $\omega = m + n$. Then

$$L_\omega = L_{m+n} = L_{m-n} + 5F_m F_n = L_2 + 5F_m F_{m-2} = 3 + 5F_m F_{m-2}.$$

But F_m and F_{m-2} are of the same order, since

$$F_m = F_{m-1} + F_{m-2} = 2F_{m-2} + F_{m-3} = F_{m-2}(2 + F_{m-3}/F_{m-2}),$$

ie $F_m/F_{m-2} = 2 + F_{m-3}/F_{m-2}$ is appreciable.

3) Concerning P_ω. We distinguish two cases:

a) $\omega = 2j$. By Koshy [25, page 123, identity 28], $P_\omega = P_{2j} = 2P_j Q_j$. Also by [25, page 123, identity 31] $2P_j^2 = -(-1)^j + Q_j^2$. We deduct that P_j and Q_j are of the same order; then also $2P_j$ and Q_j are of the same order.

b) $\omega = 2j + 1$. We see that

$$P_{2j+1} = P_{2j} + Q_{2j} = P_{2j} + 2Q_j^2 - (-1)^j = 2P_j Q_j + 2Q_j^2 - (-1)^j = Q_j(2P_j + 2Q_j) - (-1)^j,$$

where as above P_j and Q_j are of the same order.

4) Concerning Q_ω. We distinguish two cases:

a) ω is odd. In this case $Q_\omega = Q_{2\alpha+1}$. Then from [25, page 146, exercise 7, identity 1], we have $Q_\omega = P_{2\alpha+1} + P_{2\alpha} = 2P_{2\alpha} + Q_{2\alpha}$. Applying [25, page 125, identity 32] and [25, page 123, identity 28], we get

$$Q_\omega = -(-1)^\alpha + 4P_\alpha Q_\alpha + 2Q_\alpha^2 = -(-1)^\alpha + Q_\alpha(4P_\alpha + Q_\alpha),$$

where $Q_\alpha \sim 4P_\alpha + Q_\alpha$.

b) ω is even. In this case, $Q_\omega = Q_{2\alpha}$. Then from [25, page 125, identity 32] we have $Q_\omega = -(-1)^\alpha + 2Q_\alpha^2$.

5) Concerning G_ω. We distinguish two cases:

a) ω is odd. Put $m = (\omega - 1)/2 + 1$, $n = (\omega - 1)/2$. That is, $\omega = m + n$. Hence we distinguish two subcases:

a-1) n is odd. By [26, page 114, identity 30], we have

$$G_\omega = G_{m-n} + G_m L_n = G_1 + G_m L_n = a + G_m L_{m-1}.$$

Since a, b are limited and m is unlimited, G_m/F_{m-1} is appreciable because by definition we have

$$G_m = aF_{m-2} + bF_{m-1} = F_{m-1}\left(\frac{aF_{m-2}}{F_{m-1}} + b\right).$$

On the other hand, by (12), the formula $5F_{m-1}^2 + 4(-1)^{m-1} = L_{m-1}^2$ implies L_{m-1}/F_{m-1} is appreciable. Consequently, G_m/L_{m-1} is appreciable, ie G_m and L_{m-1} are of the same order.

a-2) n is even. From [26, page 114, identity 30] we also have

$$G_\omega = G_{m-n} + (G_{m+1} + G_{m-1})F_n = a + (G_{m+1} + G_{m-1})F_{m-1}.$$

As above we can prove that $(G_{m+1} + G_{m-1})/F_{m-1}$ is appreciable, ie $G_{m+1} + G_{m-1}$ and F_{m-1} are of the same order.

b) ω is even. Put $m = (\omega + 2)/2 + 1$, $n = (\omega - 2)/2$. Then $\omega = m + n$ and $m - n = 2$. By the same discussion as in the previous cases, we end the proof in question. □

Proposition 5.2 *Let ω be unlimited, and let F_ω be the ω-th Fibonacci number. If ω is even, then F_ω is of the form $\omega_1\omega_2$ where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$.*

Proof Since ω is even, we conclude that $F_\omega = F_{2m} = F_m L_m$. The result holds since by (12), L_m/F_m is appreciable. □

6 Representation of integers having limited numbers of distinct prime factors

An integer is said to be *smooth* if it is composed entirely of small prime factors. Smooth numbers play a crucial role in many interesting number theoretic and cryptography problems, such as integer factorization (see, eg, Pomerance [31]). In this context, an integer is said to be k-*smooth* if it has no prime factor greater than k. So the powers of 2 are the only 2-smooth numbers. Similarly, 3-smooth numbers are of the form $2^a 3^b$ and so on. A positive integer is said to be *limitedly smooth* if its prime factors are bounded by a limited integer. Using these numbers we formulate a generalization of a result stated in our paper [5, Proposition 2.12]. That is, we can derive a representation of some types of unlimited numbers in the form (A_2) using their factorization as product of prime factors.

Proposition 6.1 *Let $n \cong +\infty$ be limitedly smooth. Then n is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$.*

Proof Assume that $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$ where k is limited, $q_1, q_2, ..., q_k$ are limited distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k$ are unlimited positive integers. There are three cases to consider:

Case 1. $\alpha_1, \alpha_2, ..., \alpha_k$ are all even. Clearly,

$$n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k} = 1 + \left(q_1^{\frac{\alpha_1}{2}} q_2^{\frac{\alpha_2}{2}} ... q_k^{\frac{\alpha_k}{2}} - 1 \right) \left(q_1^{\frac{\alpha_1}{2}} q_2^{\frac{\alpha_2}{2}} ... q_k^{\frac{\alpha_k}{2}} + 1 \right),$$

which has the required property.

Case 2. $\alpha_1, \alpha_2, ..., \alpha_k$ are all odd. Here we see that

$$\begin{aligned} n &= q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k} = q_1 q_2 ... q_k \cdot q_1^{\alpha_1 - 1} q_2^{\alpha_2 - 1} ... q_k^{\alpha_k - 1} \\ &= q_1 q_2 ... q_k + q_1 q_2 ... q_k \left(\prod_{i=1}^{k} q_i^{\frac{\alpha_i - 1}{2}} - 1 \right) \left(\prod_{i=1}^{k} q_i^{\frac{\alpha_i - 1}{2}} + 1 \right). \end{aligned}$$

Case 3. Among the numbers $\alpha_1, \alpha_2, ..., \alpha_k$ assume that $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_s}$ are even and $\alpha_{i_{s+1}}, \alpha_{i_{s+2}}, ..., \alpha_{i_k}$ are odd for $s = 1, 2, ..., k-1$ (the number k must be ≥ 2). Therefore,

$$n = \prod_{j=s+1}^{k} q_{i_j} + \prod_{j=s+1}^{k} q_{i_j} \left(\prod_{j=1}^{s} q_{i_j}^{\frac{\alpha_{i_j}}{2}} \prod_{j=s+1}^{k} q_{i_j}^{\frac{\alpha_{i_j} - 1}{2}} - 1 \right) \left(\prod_{j=1}^{s} q_{i_j}^{\frac{\alpha_{i_j}}{2}} \prod_{j=s+1}^{k} q_{i_j}^{\frac{\alpha_{i_j} - 1}{2}} + 1 \right),$$

as desired. This completes the proof. □

Let the positive integer $\overline{a_n a_{n-1} ... a_1 a_0}$, where a_i are digits like $0, 1, ..., 9$ with $n \geq 1$ and $a_n \neq 0$. Define the function f from $\mathbb{N}$ to itself by $f(n) = \prod_{i=0}^{n}(a_i + 1)$.

Corollary 6.2 *If $f(n)$ is unlimited, then $f(n)$ is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and ω_1, ω_2 are two unlimited positive integers with $\omega_1 \sim \omega_2$.*

Proof From the definition of $f(n)$, we have $f(n) = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4}$, for some integers α_i $(1 \leq i \leq 4)$. Since $f(n)$ is unlimited, then $\max(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cong +\infty$. Thus, the result holds immediately by Proposition 6.1. □

Proposition 6.3 *Let n be an unlimited positive integer. If the number of distinct prime factors of n is limited, then*

$$\sum_{\substack{\gcd(r,n)=1 \\ 2\leq r\leq n-1}} r$$

is of the form $-1+\omega_1\omega_2$ where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$.

Proof Put $n=q_1^{\alpha_1}q_2^{\alpha_2}...q_k^{\alpha_k}$ where k is limited, $q_1,q_2,\dots,q_k$ are distinct primes and $a_1,a_2,\dots,a_k$ are positive integers. By Mollin [27, page 100], we have

$$\sum_{\substack{\gcd(r,n)=1 \\ 2\leq r\leq n-1}} r=-1+\frac{\varphi(n)n}{2},$$

where $\varphi(n)$ is the Euler's function of n. Thus n is of the form $-1+\omega_1\omega_2$ since $\varphi(n)$ is even. Moreover, we can write

$$\frac{n}{\varphi(n)}=\frac{q_1}{q_1-1}\cdot\frac{q_2}{q_2-1}\cdot\ldots\cdot\frac{q_k}{q_k-1},$$

from which it follows that $n/\varphi(n)$ is appreciable. The proof is finished. □

7 Representation of unlimited integers as in (A_k) with $k\geq 2$

First, we present some equivalent forms of (F_2) and (A_2), respectively.

Theorem 7.1 *Let n be an unlimited positive integer. Then n is of the form (F_2) if and only if n can be written as $m+w_1w_2$ where w_1,w_2 and m are integers and there exist integers x and y such that w_1+x and w_2+y are unlimited and $s-(w_1y+w_2x+xy)$ is limited.*

Proof Suppose that n can be written in the form (F_2). That is, $n=s+\omega_1\omega_2$ such that $\omega_1\cong+\infty$, $\omega_2\cong+\infty$ and s is limited. In this case, we choose $m=s$, $w_1=\omega_1$, $w_2=\omega_2$ and $x=y=0$.

Conversely, assume that $n=m+w_1w_2$, where w_1,w_2 and m are integers, and there exist integers x and y such that w_1+x and w_2+y are unlimited integers with $m-(w_1y+w_2x+xy)$ is limited. Then clearly, $s=m-(w_1y+w_2x+xy)$, $\omega_1=w_1+x$ and $\omega_2=w_2+y$, and so n can be written in the form (F_2). □

An equivalent form of (A_2) is given in the following corollary:

Corollary 7.2 *Let n be an unlimited positive integer. Then n is of the form* (A_2) *if and only if n can be written as $m + w_1w_2$ where w_1, w_2 and m are integers, and there exist integers x and y such that $w_1 + x$ and $w_2 + y$ are unlimited where $(w_1 + x)/(w_2 + y)$ is appreciable and $m - (w_1y + w_2x + xy)$ is limited.*

Let us denote by $\mathcal{A}$ the collection of all $n \in \mathbb{N}$ such that $n = s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and ω_1, $\omega_2 \in \mathbb{N}$ are unlimited with $\omega_1/\omega_2 \cong 1$. A natural question is: Are there integers m, n in $\mathcal{A}$ such that mn is also in $\mathcal{A}$?

Theorem 7.3 *There are infinitely*[3] *many unlimited positive integers $m, n \in \mathcal{A}$ such that $mn \in \mathcal{A}$.*

Proof Let a, d be two limited integers such that $(a, d) = 1$. By Lagrange's identity (Jarvis [23, Lemma 1.18, page 9]), we have

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 \tag{15}$$

for all $b, c \in \mathbb{Z}$. Let s be a limited integer. Since $\gcd(a, d) = 1$, then the equation

$$ax + yd = s \tag{16}$$

has, by transfer, a standard solution, say $(x_0, y_0) \in \mathbb{Z}^2$. Hence the general solution of (16) is represented by $x(t) = x_0 + dt$ and $y(t) = y_0 - at$ where $t \in \mathbb{Z}$. Let ω be an unlimited positive integer and put $L = \{t \in \mathbb{Z} : t \geq \omega\}$. Hence we can take $m(t) = d^2 + x^2(t)$ and $n(t) = a^2 + y^2(t)$, where t runs over L. Then $x(t)$ and $y(t)$ are simultaneously unlimited. Moreover, $m(t)$ and $n(t)$ are in $\mathcal{A}$ for each $t \in L$. By (15) and (16), the product $m(t)n(t)$ is also in $\mathcal{A}$ for each $t \in L$. □

Another aim of this section is to give some examples when a standard polynomial of degree $k \geq 3$ is of the form (A_k). Here, we use the notation: Let $n = q_1^{a_1} q_2^{a_2} ... q_r^{a_r}$ be the factorization of n as a product of powers of distinct primes, and let $set(n)$ denote the set $\{q_1, q_2, ..., q_r\}$ with $set(1) = \emptyset$. We have the following result:

Theorem 7.4 *Let $k \geq 2$ be limited. There exists a standard integer-valued polynomial $p_k(x)$ of degree k whose leading coefficient is positive such that, for all unlimited ω, $p_k(\omega)$ is of the form $-1 + \omega_1\omega_2 ... \omega_k$ where $\omega_1 \sim \omega_2 \sim ... \sim \omega_k$ and $\gcd(\omega_i, \omega_j) = 1$ for $i \neq j$.*

[3] We say that a collection of elements contains infinitely many elements of a given specificity if it contains an internal part which itself contains infinitely many of these elements.

We need the following lemma.

Lemma 7.5 *There exists a sequence of positive integers a_i $(1 \leq i \leq k)$ such that:*

- $a_1 < a_2 < ... < a_k$.
- $set(a_j - a_i) \subset set(a_j)$ *for* $1 \leq i < j \leq k$.

Proof Let $p_1 = 2, p_2 = 3, ..., p_r, ...$ be the enumeration of all primes in increasing order. Define a sequence N_s for $1 \leq s \leq k$ recursively as follows: $N_1 = 2$;

$$N_s = \prod_{1 \leq r \leq k \cdot N_{s-1}} p_r$$

for $s \geq 2$. Next, let $a_j = \sum_{\ell=k-j+1}^{k} N_\ell$. So

$$a_1 = N_k, a_2 = N_{k-1} + N_k, \ldots, a_k = N_1 + ... + N_{k-1} + N_k.$$

For $i < j$ we have

$$(17)\quad a_j - a_i = \sum_{\ell=k-j+1}^{k} N_\ell - \sum_{\ell=k-i+1}^{k} N_\ell = N_{k-(j-1)} + N_{k-(j-2)} + ... + N_{k-i} < k \cdot N_{k-i}$$

and $a_j = (a_j - a_i) + \sum_{\ell=k-i+1}^{k} N_\ell$. Now, let q be a prime that divides $a_j - a_i$. It follows from (17) that $q < k \cdot N_{k-i}$. So, if $q = p_r$ for some positive integer r then $r < k \cdot N_{k-i}$ and so this prime q divides N_ℓ for each $k - i + 1 \leq \ell \leq k$, by the definition of the sequence N_s. Hence q divides a_j. This completes the proof. □

Proof of Theorem 7.4 Let $a_1, a_2, ..., a_k$ be a sequence as in Lemma 7.5 and define the polynomial:

$$p_k(x) = a_1...a_k x^k + \sum_{a_1 \leq a_{i_1} < ... \leq a_k} a_{i_1}...a_{i_{k-1}} x^{k-1} + \sum_{a_1 \leq a_{i_1} < ... \leq a_k} a_{i_1}...a_{i_{k-2}} x^{k-2} + ... + \sum_{i=1}^{k} a_i x$$

First, we see that $p_k(\omega) = -1 + (1 + a_1\omega)(1 + a_2\omega)...(1 + a_k\omega)$, which is of the form $-1 + \omega_1\omega_2...\omega_k$, where $\omega_1 \sim \omega_2 \sim ... \sim \omega_k$ since $a_1, a_2, ..., a_k$ are limited. Let $d_{i,j} = \gcd(1 + a_i\omega, 1 + a_j\omega)$ for $1 \leq i < j \leq k$. Therefore, $d_{i,j}$ divides $(a_j - a_i)\omega$. Since $set(a_j - a_i) \subset set(a_j)$ for $1 \leq i < j \leq k$, then $d_{i,j}$ divides $a_j\omega$ for $1 \leq i < j \leq k$. Hence, $d_{i,j}$ divides 1, and so $d_{i,j} = 1$ for $1 \leq i < j \leq k$. Therefore, $\gcd(\omega_i, \omega_j) = 1$ for $i \neq j$. This completes the proof. □

As an application of Theorem 7.4 we present the following example:

Example 7.6 Let ω be unlimited. We apply the above Theorem with $a_1 = 2, a_2 = 3$ and $a_3 = 4$. Thus, $24\omega^3 + 26\omega^2 + 9\omega$ is of the form $s + \omega_1\omega_2\omega_3$ where $\omega_1 \sim \omega_2 \sim \omega_3$ and $\gcd(\omega_i, \omega_j) = 1$ for $i \neq j$. Similarly, we obtain the same result with the polynomial $216\omega^3 + 126\omega^2 + 21\omega$ for $a_1 = 3, a_2 = 6$ and $a_3 = 12$.

Remark 7.7 Let k, n be positive integers with $k \geq 2$ limited and n unlimited. Assume further that n is of the form (A_k). Then this representation is unique in the following sense: if $(\omega'_1, \omega'_2, ..., \omega'_k)$ is another k-tuple that satisfies (A_k) for some limited integer s' with ω'_i/ω'_j appreciable for $i, j = 1, 2, ..., k$, then ω'_i/ω_j is also appreciable for $i, j = 1, 2, ..., k$. We have immediately that each $\omega_i \sim \sqrt[k]{n}$ and also each $\omega'_j \sim \sqrt[k]{n}$ $(1 \leq i, j \leq k)$, hence $\omega_i \sim \omega'_j$.

Corollary 7.8 *Let n, m be two unlimited positive integers satisfying (A_k). That is, $n = s + \omega_1\omega_2...\omega_k$ and $m = s' + \omega'_1\omega'_2...\omega'_k$ where ω_i/ω_j and ω'_i/ω'_j are appreciable for $i, j = 1, 2, ..., k$. If m/n is appreciable, then ω'_i/ω_j is also appreciable for $i, j = 1, 2, ..., k$.*

Proof We have $\omega_i \sim \sqrt[k]{n}$ and $\omega'_j \sim \sqrt[k]{m}$, hence $\omega_i \sim \omega'_j$, hence

$$\omega'_j/\omega_i \sim \sqrt[k]{m}/\sqrt[k]{n} \sim m/n \qquad \square$$

Some special numbers can be represented as $1 + \omega_1\omega_2\omega_3\omega_4$, where $\omega_i \sim \omega_j$ for $1 \leq i \leq j \leq 4$.

Example 7.9 Let F_n be the n-th Fibonacci number with n unlimited. Then F_n^4 is of the form $1 + \omega_1 \cdot \omega_2 \cdot \omega_3 \cdot \omega_4$ where $\omega_1 \sim \omega_2 \sim \omega_3 \sim \omega_4$. In fact, by Andrica and Andreescu [3, Problem 9.6.3, page 359] we have $F_\omega^4 = 1 + F_{\omega-2}F_{\omega-1}F_{\omega+1}F_{\omega+2}$ where $F_{\omega+i} \sim F_{\omega+j}$ for every limited $i, j \in \mathbb{Z}$.

Example 7.10 Let q be an unlimited prime number as in Example 3.3 (see Section 3). Then the number $q^4 - 5q^2$ is of the form $-4 + \omega_1 \cdot \omega_2 \cdot \omega_3 \cdot \omega_4$, where $\omega_1 \sim \omega_2 \sim \omega_3 \sim \omega_4$ with $\gcd(\omega_i, \omega_j) = 1$ for $i \neq j$. Indeed, we see that

$$q^4 - 5q^2 = -4 + (q-2)(q-1)(q+1)(q+2) = -4 + s_1\pi_1 \cdot s_2\pi_2 \cdot s_3\pi_3 \cdot s_4\pi_4$$

where $s_i \in \mathbb{N}$ is limited and π_i is unlimited prime for $1 \leq i \leq 4$. We let $\omega_1 = s_1s_2s_3s_4\pi_1$, $\omega_2 = \pi_2$, $\omega_3 = \pi_3$ and $\omega_4 = \pi_4$.

We give some examples of standard integer-valued polynomials $f(x)$ of odd degree k whose leading coefficient is positive, such that $f(\omega)$ for ω unlimited cannot be written as

$$f(\omega) = s + \omega_1\omega_2...\omega_k, \tag{18}$$

where $s \in \mathbb{Z}$ is limited and $\omega_i = a_i\omega + x_i \in \mathbb{N}$ with $a_i, x_i \in \mathbb{Q}^*$ limited for all $1 \leq i \leq k$. Let us start with a polynomial of degree 3 that cannot be written as in (18).

Proposition 7.11 *Let $\omega \in \mathbb{N}$ be unlimited. The natural number $\omega(\omega^2 + \omega + 1)$ is not of the form $s + (a_1\omega - x_1)(a_2\omega - x_2)(a_3\omega - x_3)$ where $a_i, x_i \in \mathbb{Q}^*$ are limited for $1 \leq i \leq 3$ with $s = x_1x_2x_3$.*

Proof Suppose, by way of contradiction, that

$$\omega(\omega^2 + \omega + 1) = s + (a_1\omega - x_1)(a_2\omega - x_2)(a_3\omega - x_3)$$

for some limited rational numbers a_i, x_i $(1 \leq i \leq 3)$ with $s = x_1x_2x_3 \in \mathbb{Z}$. Since ω is unlimited, then $a_1a_2a_3 = 1$ and $-a_1a_2x_3 - a_1a_3x_2 - a_2a_3x_1 = 1$. It follows from these equations that $x_1 = -a_1 - a_1^2a_2x_3 - a_1^2a_3x_2$, and since $a_1x_2x_3 + a_2x_1x_3 + a_3x_1x_2 = 1$ we obtain

$$-a_1^2a_2^2 \cdot x_3^2 + (a_1x_2 + a_1a_2)x_3 + a_1^2a_3^2x_2^2 + a_1a_3x_2 + 1 = 0. \tag{19}$$

Now, assume that $(\bar{x}_2, x_3)$ is a solutions of the equation (19). That is,

$$a_1^2a_2^2 \cdot x_3^2 + (a_1\bar{x}_2 + a_1a_2)x_3 + a_1^2a_3^2\bar{x}_2^2 + a_1a_3\bar{x}_2 + 1 = 0.$$

The above equation has no rational solutions since its discriminant

$$\Delta = -3a_1^2a_2^2 - 2a_1^2a_2\bar{x}_2 - 3a_1^2\bar{x}_2^2$$

is negative. In fact, we distinguish two cases:

Case1. Assume that a_2 and $\bar{x}_2$ have the same signs. In this case we have $\Delta < 0$.

Case 2. Assume that a_2 and $\bar{x}_2$ have different signs. Then clearly $a_2\bar{x}_2 < 0$, and hence

$$3a_1^2a_2^2 + 2a_1^2a_2\bar{x}_2 + 3a_1^2\bar{x}_2^2 = \begin{cases} 3a_1^2((a_2')^2 + \bar{x}_2^2) - 2a_1^2(a_2'\bar{x}_2), \text{ if } a_2 = -a_2' < 0 \\ 3a_1^2(a_2^2 + (x_2')^2) - 2a_1^2(a_2x_2'), \text{ if } \bar{x}_2 = -x_2' < 0. \end{cases}$$

Since $(a_2')^2 + \bar{x}_2^2 > a_2'\bar{x}_2$ and $a_2^2 + (x_2')^2 > a_2x_2'$, we conclude that $\Delta < 0$. □

Proposition 7.12 *Let $k \geq 3$ be odd limited, and let ω be an unlimited positive integer. The natural number $n = \omega^k$ is not of the form $s + (a_1\omega - x_1)(a_2\omega - x_2)...(a_k\omega - x_k)$ where $a_1, ..., a_k \in \mathbb{Q}^*$ are limited with $a_k \in]-1, 1[$, $x_1, ..., x_k \in \mathbb{Q}^*$ are limited and $s = x_1x_2...x_k$.*

Proof When $k = 3$. Assume that $\omega^3 = s + (a_1\omega - x_1)(a_2\omega - x_2)(a_3\omega - x_3)$, where $a_i, x_i \in \mathbb{Q}^*$ are limited for $1 \leq i \leq 3$ and $s = x_1x_2x_3$. Applying (22), we obtain $-a_1^2a_2^2z^2 + (-2a_1x_2 + a_1x_2)z - a_1^2a_3^2x_2^2 = 0$, where $z = x_3$. The discriminant of this equation is $\Delta = -3a_1^2x_2^2 < 0$, and hence there are no positive rational solutions x_1, x_2, x_3 of the above equation.

Now, let $k \geq 5$ and assume that $\omega^k = s + (a_1\omega - x_1)(a_2\omega - x_2)...(a_k\omega - x_k)$, where $a_1, ..., a_k, x_1, ..., x_k \in \mathbb{Q}^*$ are limited and $s = x_1x_2...x_k$. Then $a_1a_2...a_k = 1$ and

$$0 = \sum_{i=1}^{k} a_1...a_{i-1}a_{i+1}...a_kx_i, \tag{20}$$

$$0 = \sum_{1\leq i<j\leq k} a_1...a_{i-1}a_{i+1}...a_{j-1}a_{j+1}...a_kx_ix_j. \tag{21}$$

It follows from (20) that $x_1 = \sum_{i=2}^{k} a_1^2...a_{i-1}a_{i+1}...a_kx_i$, and by (21) we have

$$\begin{aligned} &-\left(\prod_{i=1}^{k-1} a_i^2\right) x_k^2 - 2\left[\sum_{2\leq i\leq k-1} a_1...a_{i-1}a_{i+1}...a_kx_i\right] x_k \\ &-\sum_{i=2}^{k-1} a_1^2...a_{i-1}^2a_{i+1}^2...a_k^2x_i^2 - 2\sum_{2\leq i<j\leq k-1} a_1...a_{i-1}a_{i+1}...a_{j-1}a_{j+1}...a_kx_ix_j = 0. \end{aligned} \tag{22}$$

This quadratic equation has the following discriminant:

$$\begin{aligned} \Delta &= \left(\sum_{2\leq i\leq k-1} a_1...a_{i-1}a_{i+1}...a_kx_i\right)^2 + \left(\prod_{i=1}^{k-1} a_i^2\right) \times \\ &\quad \left(-\sum_{i=2}^{k-1} a_1^2...a_{i-1}^2a_{i+1}^2...a_{k-1}^2x_i^2 - 2\sum_{2\leq i<j\leq k-1} a_1...a_{i-1}a_{i+1}...a_{j-1}a_{j+1}...a_{k-1}x_ix_j\right) \\ &= -\sum_{i=2}^{k-1} a_1^4...a_{i-1}^4a_i^2a_{i+1}^4...a_{k-1}^4x_i^2 + \sum_{i=2}^{k-1} a_1^2...a_{i-1}^2a_{i+1}^2...a_{k-1}^2x_i^2 - \\ &\quad 2\sum_{2\leq i<j\leq k-1} a_1^3...a_{i-1}^3a_i^2a_{i+1}^3...a_{j-1}^3a_j^2a_{j+1}^3...a_{k-1}^3x_ix_j + \\ &\quad 2\sum_{2\leq i<j\leq k-1} a_1^2...a_{i-1}^2a_ia_{i+1}^2...a_{j-1}^2a_ja_{j+1}^2...a_{k-1}^2x_ix_j. \end{aligned}$$

Since $a_1a_2...a_k = 1$ and $a_k^2 < 1$, then $a_1^2...a_{i-1}^2a_{i+1}^2...a_{k-1}^2 < a_1^4...a_{i-1}^4a_i^2a_{i+1}^4...a_{k-1}^4$ and so

$$a_1^2...a_{i-1}^2a_{i+1}^2...a_{k-1}^2x_i^2 < a_1^4...a_{i-1}^4a_i^2a_{i+1}^4...a_{k-1}^4x_i^2, \text{ for } i = 2, ..., k-1.$$

Similarly, we have

$$a_1^2...a_{i-1}^2a_ia_{i+1}^2...a_{j-1}^2a_ja_{j+1}^2...a_{k-1}^2 < a_1^3...a_{i-1}^3a_i^2a_{i+1}^3...a_{j-1}^3a_j^2a_{j+1}^3...a_{k-1}^3.$$

Since x_i and x_j have the same parity, it follows that

$$a_1^2 \dots a_{i-1}^2 a_i a_{i+1}^2 \dots a_{j-1}^2 a_j a_{j+1}^2 \dots a_k^2 x_i x_j < a_1^3 \dots a_{i-1}^3 a_i^2 a_{i+1}^3 \dots a_{j-1}^3 a_j^2 a_{j+1}^3 \dots a_k^3 x_i x_j.$$

Thus, we have shown that $\Delta < 0$, and hence (22) has no rational solutions. The proof is finished. □

8 Examples of the natural numbers of the form (AR_2)

In this section we present some families of unlimited positive integers which can be written as the sum of a limited integer and the product of two relatively prime unlimited positive integers having the same order. That is, we present examples in which $\gcd(\omega_1, \omega_2) = 1$ and ω_1/ω_2 is appreciable.

Now, let q be a limited prime number. Is q^n of the form $1 + \omega_1\omega_2$ and $-1 + \varpi_1\varpi_2$ where $\omega_1 \sim \omega_2$, $\varpi_1 \sim \varpi_2$, $\gcd(\omega_1, \omega_2) = 1$ and $\gcd(\varpi_1, \varpi_2) = 1$ for some unlimited n? When $q = 2$, the answer is positive as shown in the following proposition; but replacing the prime 2 by an odd limited prime number yields an immediate open question.

Proposition 8.1 *Let ω be unlimited. Then $2^{4\omega+2}$ can be written in the two forms: $1 + \omega_1\omega_2$ and $-1 + \varpi_1\varpi_2$ where $\omega_1 \sim \omega_2$, $\varpi_1 \sim \varpi_2$, $\gcd(\omega_1, \omega_2) = 1$ and $\gcd(\varpi_1, \varpi_2) = 1$.*

Proof Since $2^{4\omega+2} + 1 = 4(2^{2\omega})^2 + 1$, it follows that

$$2^{4\omega+2} = -1 + (2^{2\omega+1} + 2^{\omega+1} + 1)(2^{2\omega+1} - 2^{\omega+1} + 1).$$

The two factors are both odd, and their difference is $2^{\omega+2}$; hence, they are relatively prime. On the other hand, it is clear that $2^{4\omega+2} = 1 + (2^{2\omega+1} - 1)(2^{2\omega+1} + 1)$. This completes the proof. □

Example 8.2 Let ω be an odd unlimited integer, and let $n = \omega^4 + 2^{2\omega}$. One can prove that n is the form $\omega_1\omega_2$ where $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. Indeed, we see that

$$\begin{aligned} n &= \omega^4 + 2^{2\omega} = \omega^4 + 2\omega^2 \cdot 2^\omega + 2^{2\omega} - 2\omega^2 \cdot 2^\omega \\ &= (\omega^2 + 2^\omega)^2 - (\omega \cdot 2^{\frac{\omega+1}{2}})^2 = (\omega^2 + 2^\omega + \omega \cdot 2^{\frac{\omega+1}{2}})(\omega^2 + 2^\omega - \omega \cdot 2^{\frac{\omega+1}{2}}). \end{aligned}$$

We put $\omega_1 = \omega^2 + 2^\omega + \omega \cdot 2^{(\omega+1)/2}$ and $\omega_1 = \omega^2 + 2^\omega - \omega \cdot 2^{(\omega+1)/2}$. Therefore, $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$.

Next, we present some examples on polynomials with integer coefficients, which can be written in the form (AR_k). In particular, we ask whether a polynomial of degree 2 of the form $a\omega + b\omega^2$ where $a, b \neq 0$ are limited can be expressed as in (A_2).

Proposition 8.3 *Let $n = \omega^2 + b\omega$, where $b \in \mathbb{Z}$ is limited. Then n is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$, $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$.*

Proof First, if $b = 0$ then

$$n = \omega^2 = \begin{cases} 1 + (\omega - 1)(\omega + 1), \text{ if } \omega \text{ is even} \\ 4 + (\omega - 2)(\omega + 2), \text{ otherwise.} \end{cases} \tag{23}$$

Next, assume that b is even and set $b = 2^k m$ with $(m, 2) = 1$. There are two cases:

- $k = 1$ and ω is odd or $k \geq 2$ and ω is even. We see that
$$n = 1 - \frac{b^2}{4} + \left(\omega + \frac{b}{2} - 1\right)\left(\omega + \frac{b}{2} + 1\right).$$
- $k = 1$ and ω is even or $k \geq 2$ and ω is odd. We sse that
$$n = \frac{16 - b^2}{4} + \left(\omega + \frac{b}{2} - 2\right)\left(\omega + \frac{b}{2} + 2\right).$$

Now, if b is odd we have
$$n = \frac{1 - b^2}{4} + \left(\omega + \frac{b-1}{2}\right)\left(\omega + \frac{b-1}{2} + 1\right).$$
Thus, in all cases, n is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited, $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. This completes the proof. □

Corollary 8.4 *Let ω be unlimited and let $a \geq 1$ be limited. Then the number $n = a\omega^2 + a\omega$ is of the form $s + \omega_1\omega_2$ where $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$.*

Proof Applying Proposition 8.3, $\omega^2 + \omega$ is of the form $s + w_1w_2$ where $w_1 \sim w_2$ and $\gcd(w_1, w_2) = 1$. Since a is limited, the result follows immediately by distributing the prime powers that divide a over those of w_1 and w_2 so that $a \cdot w_1w_2 = \omega_1\omega_2$ where $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. □

Example 8.5 Let q be an unlimited prime number of the form $4k + 1$ or $8k \pm 1$. There are infinitely many n such that $n \cdot q$ is of the form $s + \omega_1\omega_2$ where $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In fact, if q is of the form $4k + 1$, then by Adler and Coury [1, Theorem 5.11, page 130], there exists an integer $\widetilde{x}$ such that

$$\widetilde{x}^2 = -1 + \lambda_0 q \tag{24}$$

for some positive integer λ_0. Since $\widetilde{x}+\lambda q$ also satisfies (24) for every $\lambda \geq 1$, there exist infinitely many n such that $n \cdot q = 1 + x^2$, where $x \in \mathbb{N}$ is unlimited. Similarly, by [1, Theorem 5.12, page 130], if q is of the form $8k \pm 1$ then there exist infinitely many n such that $n \cdot q = -2 + x^2$ for some unlimited $x \in \mathbb{N}$. In both cases, by applying (23) we have the desired result.

9 Unlimited natural numbers of the form $s + \omega^k$

Let us start with an example involving Lucas numbers (L_ω) with $\omega \cong +\infty$. From Koshy [26, page 97, Formula 41], if ω is even then $L_{2\omega} = -2 + L_\omega^2$ and if ω is odd, then $L_{2\omega} = 2 + L_\omega^2$.

Many conjectures in elementary number theory lead to the existence of unlimited prime numbers of the form $s + \omega^k$ where s and k are limited. Let us mention two conjectures.

Conjecture 9.1 (see Shanks [36, page 31]) *There are infinitely many primes of the form* $-2 + \omega^2$.

Conjecture 9.2 (see Nathanson [28, page 287]) *There are infinitely many primes of the form* $1 + \omega^2$.

In the same way, we get:

Proposition 9.3 *Let n be unlimited. Then n^2 is not of the form $s + \omega^2$ where $s \in \mathbb{Z}^*$ is limited and $\omega \in \mathbb{N}$ is unlimited.*

Proof If n^2 can be written in the desired form, then $(n-\omega)(n+\omega) = s$. But, this is a contradiction since the left-hand side of this equation is unlimited, while its right-hand side is limited. □

We can deduce from Proposition 9.3 the following result.

Corollary 9.4 *Let n be unlimited odd and let $s \in \mathbb{Z}^*$ be limited. Then $n-s$ and $n+s$ are not both a perfect square.*

In addition, assume that $s \in \mathbb{Z}^*$ is odd (limited or not). There are no positive integers n such that $n-s$ and $n+s$ are a perfect square. Otherwise, $(u-v)(u+v) = 2s$ for some positive integers u, v. Since $u-v$ and $u+v$ are distinct and of the same parity, the only possibility is $u-v = 2$ and $u+v = s$. Thus, s is even and this is a contradiction.

The following facts are easy consequences of results in the literature:

(1) Erdös and Selfridge [15] proved that a product of consecutive integers can never be a perfect power. In this context, the product of two consecutive unlimited integers cannot be written in the form $s+\omega^2$, where $s \in \mathbb{Z}$ is limited and $\omega \in \mathbb{N}$ is unlimited. In fact, if $n(n+1) = s+\omega^2$, then clearly $\omega > n$ and $(\omega - n)(\omega + n) = n - s$ which is impossible.
(2) Let n be an unlimited positive integer of the form $4k+1$. From Mollin [27, Example 1.26, page 51], there does not exist an unlimited integer λ such that $\lambda \cdot n = 1+\omega^2$ for some unlimited integer ω.
(3) Let n be unlimited. Assume that $n = R \cdot S$ where the canonical factorization of R (resp. of S) is $2^{\alpha} \prod_{p\equiv 1 (\mathrm{mod}\ 4)} p^{\beta} \prod_{q\equiv 3 (\mathrm{mod}\ 4)} q^{\gamma}$ (resp. $2^{\widetilde{\alpha}} \prod_{p\equiv 1 (\mathrm{mod}\ 4)} p^{\widetilde{\beta}} \prod_{q\equiv 3 (\mathrm{mod}\ 4)} q^{\widetilde{\gamma}}$) with $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta} \geq 0$ and $\max(\gamma, \widetilde{\gamma}) \geq 1$. If $\sum \gamma + \sum \widetilde{\gamma}$ is odd, then n cannot be written as $1+\omega^2$. Indeed, by Niven [30, Theorem 2.15, page 55], at least one of the integers R, S cannot be written as the sum of two squares, since there is at least an exponents γ in R or an exponents $\widetilde{\gamma}$ in S which is odd. Consequently, by Fine [17, Theorem 3.2.3, page 98], n cannot be written as the sum of two squares.

Now, we deal with *Bachet's equation* which is of the form $x^3 = k + y^2$ where k is an integer. For details, see Felgner [16]. These equations have long been investigated by many authors, but for some of k, even small, not all the solutions in integers x, y have been found. We consider the case when x and y are unlimited and k is limited. We prove the following theorem:

Theorem 9.5 *There are no unlimited positive integer n such that*

$$n^3 = s^3 + \omega^2, \tag{25}$$

where $s \in \mathbb{N}^*$ is limited and $\omega \in \mathbb{N}$ has only unlimited prime factors.

Proof Assume that (9.5) holds for some unlimited $n \in \mathbb{N}$. We rewrite this equation in the form $(n-s)(n^2+ns+s^2) = \omega^2$. Note that if $\gcd(n-s, n^2+ns+s^2) = d > 1$, then d divides the linear combination $n^2+ns+s^2-(an+b)(n-s)$ where $a, b \in \mathbb{Z}$. In particular, for $a=1$ and $b=2s$ we have d divides $3s^2$. Since d divides ω^2, we conclude that there exists a limited prime number p which divides ω. This is impossible because ω has only unlimited prime factors. Therefore, both $n-s$ and n^2+ns+s^2 are perfect square. If we put $x = \sqrt{n-s}$, it follows that

$$n^2+ns+s^2 = (x^2+s)^2 + (x^2+s)s + s^2 = x^4 + 3sx^2 + 3s^2,$$

and so

$$4(n^2+ns+s^2) = (2x^2+3s)^2 + 3s^2. \tag{26}$$

We set $u = 2\sqrt{n^2 + ns + s^2}$ and $v = 2x^2 + 3s = 2n + s$, we see from (26) that

$$(u - v)(u + v) = 3s^2,$$

which contradicts the fact that $s \neq 0$ is limited and $u + v \in \mathbb{N}$ is unlimited. This completes the proof. □

Corollary 9.6 *Let $k \geq 1$ be limited. There are no unlimited positive integer n such that $n^{2k} = s^{2k} + \omega^2$ where $s \in \mathbb{N}^*$ is limited and $\omega \in \mathbb{N}$ has only unlimited prime factors.*

Proof Let n be unlimited, and assume that $n^{2k} = s^{2k} + \omega^2$, where $s \in \mathbb{N}^*$ is limited and $\omega \in \mathbb{N}$ is unlimited. We put $n^k = m$ and $s^k = s'$. Hence

$$n^{2k} - s^{2k} = (m - s')(m + s') = \omega^2.$$

Clearly, $\gcd(m - s', m + s') = 1$ since ω has only unlimited prime factors. Thus, $m + s'$ and $m - s'$ are perfect squares. We put $u = \sqrt{m + s'}$ and $v = \sqrt{m - s'}$, then $(u - v)(u + v) = 2s'$. A contradiction since $s' \in \mathbb{N}^*$ is limited and $u + v \cong +\infty$. The proof is finished. □

Corollary 9.7 *Let n be unlimited. Then n^2 is not of the form $s^2 + \omega^k$, where $s \in \mathbb{N}^*$ is limited, $k \geq 3$ is odd limited and $\omega \in \mathbb{N}$ has only unlimited prime factors.*

Proof Assume that $(n - s)(n + s) = \omega^k$ for some limited $s \in \mathbb{N}^*$, $k \geq 3$ is odd and $\omega \in \mathbb{N}$ has only unlimited prime factors. Let $\gcd(n - s, n + s) = d$. Since d divides ω^k, then $d = 1$ and so $n - s = \omega^{k_1}$ and $n + s = \omega^{k_2}$, where k_1 and k_2 are positive integers ($k_1 < k_2$ and $k_1 + k_2 = k$) one of which is odd and the other is even. Thus, we see that $n - s \not\sim n + s$. This is impossible. □

Proposition 9.8 *Let ℓ, n, ω be positive integers satisfying*

(1) *$\ell \in \mathbb{Z}^*$ is limited with $\ell \mid n$,*
(2) *n is unlimited and $n - \ell$ is prime,*
(3) *ω is unlimited, $n \mid (\omega - \ell)$, $(n - \ell) \nmid \omega$ and $\omega \mid (n^3 - \ell^3)$.*

Then ω is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and ω_1, ω_2 are two unlimited positive integers satisfying the conditions $\omega_1/\omega_2 \cong \ell$.

As an example, let us take $\ell = 1$ and $\omega = n^2 + n + 1$ where $n \cong +\infty$ and $n - 1$ is prime.

Proof of Proposition 9.8 Since $n^3 - \ell^3 = (n-\ell)(n^2+\ell n+\ell^2)$ and $\omega \mid (n^3-\ell^3)$, it follows that $\omega \mid (n^2+\ell n+\ell^2)$. Then there exists a positive integer k such that

$$k\omega = n^2 + \ell n + \ell^2. \tag{27}$$

On the other hand, $n \mid (\omega - \ell)$ implies $\omega \equiv \ell (\text{mod } n)$, and therefore $k\omega \equiv k\ell(\text{mod } n)$. By (27), $n^2+\ell n+\ell^2 \equiv k\ell(\text{mod } n)$, or equivalently, $k\ell \equiv \ell^2(\text{mod } n)$. Thus, $k \equiv \ell(\text{mod } \frac{n}{\ell})$. Therefore, $k = \ell + a\frac{n}{\ell}$, $\omega = \ell + bn$ for some integers $a \geq 0$ and $b \geq 1$ since $\omega \cong +\infty$. Substituting these values of k and ℓ in (27), we obtain $(\ell + a\frac{n}{\ell})(\ell+bn) = n^2+\ell n+\ell^2$, and hence

$$ab\frac{n}{\ell} + a + b\ell = n + \ell. \tag{28}$$

We distinguish the following cases:

1. If $\ell > 0$, then $ab\frac{n}{\ell} + a + b\ell \geq n + 1 + \ell > n + \ell$, which is impossible.

2. If $\ell < 0$, then (28) gives

$$\left(\frac{ab}{\ell'} + 1\right) n = (b-1)\ell' + a, \tag{29}$$

where $\ell' = -\ell > 0$. We also distinguish two cases:

2.1. If a, b are limited, then (29) gives $ab = -\ell' = \ell < 0$ which is not valid.

2.2. Let a or b be unlimited. In the case when $a \geq 1$ is limited, it follows from (29) that $(b-1)\ell' + a/(\frac{ab}{\ell'}+1) \cong +\infty$. A contradiction. In the case when a is unlimited and b is limited, by (29) we see that $b/\ell' \cong 0$, which is impossible. Also, if $a, b \cong +\infty$, then (28) does not hold.

Finally, we can deduce that $a = 0$ and so $k = l$. Using (27), we see that

$$\omega = \frac{n^2+\ell n+\ell^2}{\ell} = \ell + n\left(1 + \frac{n}{\ell}\right),$$

which is of the desired form where $\omega_1 = n$ and $\omega_2 = 1 + \dfrac{n}{\ell}$. □

10 Special numbers of the form (A$_2$)

A Cullen number is a number of the form $1 + n \cdot 2^n$ (denoted by C_n), where n is a nonnegative integer (see, eg, [19, B 20]). By definition, it is clear that if n is unlimited then C_n is of the form $1 + \omega_1\omega_2$ where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited; but $\omega_1 \not\sim \omega_2$. In addition, we see that $\gcd(\omega_1, \omega_2) = 1$ whenever n is odd.

The starting point for this note is the following result.

Proposition 10.1 *Let n be unlimited. Then C_n is of the form $1+\omega_1\omega_2$ where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$.*

Proof Clearly, $C_n=1+n\cdot 2^n$, where $n/2^n\cong 0$. Put $u_i=n\cdot 2^i/2^{n-i}$ for $i=0,1,...,n$. Then for every $i=0,1,...,n-1$, we have

$$\frac{u_{i+1}}{u_i}=\frac{n\cdot 2^{i+1}}{2^{n-(i+1)}}\cdot\frac{2^{n-i}}{n\cdot 2^i}=4.$$

On the other hand, we see that $u_0\cong 0$ and $u_n=n\cdot 2^n\cong+\infty$. Consequently, there exists an unlimited integer $i_0<n$ such that the number $u_{i_0}=n\cdot 2^{i_0}/2^{n-i_0}$ is appreciable. Therefore, $C_n=1+n\cdot 2^n=1+(n\cdot 2^{i_0})\cdot 2^{n-i_0}=1+\omega_1\omega_2$ where $\omega_1=n\cdot 2^{i_0}$ and $\omega_2=2^{n-i_0}$ are two unlimited positive integers such that $\omega_1\sim\omega_2$. This completes the proof. □

Remark 10.2 The previous reasoning works with any sequence of positive integers of the form $a+n\cdot b^n$ with $n\geq 1$, where a,b are limited positive integers with $b\geq 2$. For example, the generalized Cullen numbers which are numbers of the form $C_{n,s}=1+n\cdot s^n$, where $n\geq 1$ and $s\geq 2$ is limited.

Corollary 10.3 *Let $k\geq 1$ be limited and let n be unlimited such that $n=q_1^{a_1}...q_k^{a_k}$, where $q_1,...,q_k$ are limited distinct primes and $a_1,\dots,a_k$ are positive integers with $\max(a_1,\dots,a_k)\cong+\infty$. Then C_n is of the form $s+\omega_1\omega_2$ where $s\in\mathbb{N}$ is limited and $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$ and $\gcd(\omega_1,\omega_2)=2$.*

Proof This follows immediately from Proposition 6.1 since

$$C_n=1+q_1^{a_1}...q_k^{a_k}\cdot 2^n,$$

which is of the form $1+a\cdot m$, where a is limited and m is an unlimited smooth number. □

Corollary 10.4 *Let a,k,s be limited positive integers with $s\geq 2$. Define $C_{n,s,k}=a+p_k(n)\cdot s^n$, where $n\geq 1$ and p_k is an integer-valued polynomial whose leading coefficient is positive. If n is unlimited, then $C_{n,s,k}$ is of the form $a+\omega_1\omega_2$, where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$.*

Proof The proof follows the idea of the proof of Proposition 10.1. □

Remark 10.5 Let n be unlimited. We do not know whether $n\cdot 2^n$ is of the form $\omega_1\omega_2$ where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$ and $\gcd(\omega_1,\omega_2)=1$ (The proof of Proposition 10.1 does not apply).

Some numbers of the form $a \cdot p^n$ where $a, p \geq 1$ are limited and n is unlimited can be written as a sum of a nonzero limited integer and a product of two relatively prime unlimited positive integers having the same order. It is in fact the subject of Proposition 10.6. In addition, for every limited prime number q different from p, there exists an unlimited integer ω for which p^n/q^ω is appreciable, where $\gcd(p^n, q^\omega) = 1$. In the following proposition we show that $a' \cdot p^n$ can also be written as a product of two relatively prime unlimited positive integers having the same order for infinitely many positive integers a'.

Proposition 10.6 *Let $p \geq 3$ be a limited prime number and let n be unlimited. There exist infinitely many positive integers a' such that $a' \cdot p^n$ can be written in the form $\omega_1\omega_2$ where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$.*

Proof Let $\varphi(n)$ be the Euler's function of n and let q be a prime number with $p \neq q$. Since $\varphi(p^n) = (p-1)p^{n-1}$, we conclude from Euler's Theorem (see, eg, Andrica and Andreescu [3, page 147]) that $q^{(p-1)p^{n-1}} \equiv 1 (\mathrm{mod}\ p^n)$. Thus, there exists an unlimited positive integer $a_q = a'$ such that

$$a' \cdot p^n = -1 + q^{(p-1)p^{n-1}} = (q^{\frac{(p-1)p^{n-1}}{2}} - 1)(q^{\frac{(p-1)p^{n-1}}{2}} + 1).$$

Clearly, 1 is the highest power of 2 which divides both $q^{\frac{(p-1)p^{n-1}}{2}} - 1$ and $q^{\frac{(p-1)p^{n-1}}{2}} + 1$. So by Lemma 4.4, $a' \cdot p^n$ can be written as the desired form. The proof is finished. □

We now consider some additional special kinds of numbers.

(1) Let q be an odd prime. Numbers of the form $M_q = 2^q - 1$ are called Mersenne numbers[4]. If $q \cong +\infty$, then M_q is of the form $1 + \omega_1\omega_2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In fact, we have

$$M_q = 1 + 2(2^{(q-1)/2} - 1)(2^{(q-1)/2} + 1).$$

(2) Recall that the integer $f_n = 2^{2^n} + 1$ is called the n-th Fermat number. If n is unlimited, then f_n is of the form $2 + \omega_1\omega_2$ where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In fact, we have

$$f_n = 1 + (f_{n-1} - 1)^2 = 2 + f_{n-1}(f_{n-1} - 2),$$

where $\gcd(f_{n-1}, f_{n-1} - 2) = 1$.

[4]See Guy [19, page 13]; it is an unsolved problem to determine whether there are infinitely many Mersenne primes.

(3) $f_{n+1}+f_n$ is of form $1+\omega_1\omega_2$, where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$ and $\gcd(\omega_1,\omega_2)=1$. In fact, we have

$$f_{n+1}+f_n=1+(2^{2^n}+1-2^{2^{n-1}})(2^{2^n}+1+2^{2^{n-1}}).$$

(4) Let F_n be the n-th Fibonacci number. If n is unlimited, then F_n is of the form $-1+\omega_1\omega_2$, where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$ and also F_n is of the form $1+\varpi_1\varpi_2$ where $\varpi_1,\varpi_2\in\mathbb{N}$ are unlimited with $\varpi_1\sim\varpi_2$. Indeed, the claim follows immediately from Koshy [26, page 205, Theorem 16.9] since we have

$$\begin{array}{ll} \text{(a) } F_{4n}=-1+F_{2n-1}\cdot L_{2n+1} & \text{(e) } F_{4n}=1+F_{2n+1}\cdot L_{2n-1} \\ \text{(b) } F_{4n+1}=-1+F_{2n+1}\cdot L_{2n} & \text{(f) } F_{4n+1}=1+F_{2n}\cdot L_{2n+1} \\ \text{(c) } F_{4n+2}=-1+F_{2n+2}\cdot L_{2n} & \text{(g) } F_{4n+2}=1+F_{2n}\cdot L_{2n+2} \\ \text{(d) } F_{4n+3}=-1+F_{2n+1}\cdot L_{2n+2} & \text{(h) } F_{4n+3}=1+F_{2n+2}\cdot L_{2n+1}. \end{array} \tag{30}$$

(5) Let q be a Hrbáček prime number, that is, a prime number such that for every limited $s\in\mathbb{Z}$ we have $q+s=s'\cdot\pi$ for some limited $s'\in\mathbb{N}$ and for some unlimited prime number π. For each such prime q we put $n=q^2$. Then n is of the form $1+s\cdot(\pi_1\cdot\pi_2)$, where π_1,π_2 are unlimited primes with $\pi_1\sim\pi_2$. Indeed, $n=1+(q-1)(q+1)=1+(s_1\cdot\pi_1)(s_2\cdot\pi_2)$, as required.

(6) There exist infinitely many pairs of different positive integers m and n such that: (*i*) m and n have the same prime divisors. Also $n+1$ and $m+1$ have the same prime divisors as $n+1=(m+1)^2$. (*ii*) Each of the numbers $n,m,n+1$ and $m+1$ can be written in the form (A_2). For the proof it suffices to take the numbers $n=2^{\omega+1}(2^{\omega-1}-1)$ and $m=-2+2^{\omega}$ where ω is unlimited.

Proposition 10.7 *Let ω be unlimited. If ω is even, then F_ω is of the form $\omega_1\omega_2$ where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$.*

Proof Since ω is even, then by [26, page 96, Formula 29], $F_\omega=F_{2m}=F_m\cdot L_m$. The result holds since by (12) $F_m\sim L_m$. □

Let $\alpha\in\mathbb{R}_+^*$ be an appreciable real number such that its standard part α° is irrational (for details on the number α°, see Diener and Reeb [14, page 28]).

Theorem 10.8 *There exist unlimited positive integers N,ω_1,ω_2 and a limited integer $s\in\{0,1,-1\}$ such that $[N\alpha]=s+\omega_1\omega_2$ where $\omega_1\sim\omega_2$ and $\gcd(\omega_1,\omega_2)=1$.*

Proof We put $\alpha=\alpha^\circ+\theta$, where $\theta\cong0$. Let $(p_k/q_k)_{k\geq0}$ be the k-th convergent of the continued fraction of α°. Since the standard sequence of positive integers $(q_k^2)_{k\geq0}$

is monotonically increasing there exist, by Cauchy's principle, an unlimited positive integer k_0 satisfying $q_{k_0}^2\theta \cong 0$. Let k be an unlimited positive integer such that $2k < k_0$, that is, $q_{2k}^2\theta \cong 0$. On the other hand, by Hardy and Wright [20, Theorem 164, page 176], we conclude that $0 < \alpha^\circ - p_{2k}/q_{2k} < 1/q_{2k}^2$ for $k \geq 1$, from which it follows that $0 < q_{2k}^2\alpha - q_{2k}p_{2k} < 1$. Hence, $q_{2k}^2\alpha^\circ - q_{2k}p_{2k} = e$ with $0 < e < 1$. Equivalently, $q_{2k}^2\alpha^\circ = e + q_{2k}p_{2k}$. Therefore,

$$q_{2k}^2\alpha = q_{2k}^2(\alpha^\circ + \theta) = q_{2k}p_{2k} + e + q_{2k}^2\theta = q_{2k}p_{2k} + e + \widetilde{\theta},$$

where $\widetilde{\theta} \cong 0$. From above we have $\left[q_{2k}^2\alpha\right] = s + q_{2k}p_{2k}$ where $s \in \{-1, 0, 1\}$. We finish the proof if we take $N = q_{2k}^2$, $\omega_1 = q_{2k}$ and $\omega_2 = p_{2k}$. Note that $\omega_1 \sim \omega_2$, and by the properties of the continued fraction of α° we have $\gcd(\omega_1, \omega_2) = 1$. □

We give examples of numbers of the form $\omega_1\omega_2$ where ω_1, ω_2 are two unlimited positive integers such that $\omega_1 \sim \omega_2$, but they can also written in the form $s + \varpi_1\varpi_2$ where $s \in \mathbb{Z}^*$ is limited and $\varpi_1, \varpi_2 \in \mathbb{N}$ are unlimited with $\varpi_1 \sim \varpi_2$.

Example 10.9 Let ω be unlimited.

(1) From Proposition 10.7 and Equations (a) and (c) in (30), we obtain

$$F_\omega \cdot L_\omega = F_{2\omega} = \begin{cases} -1 + F_{2m-1} \cdot L_{2m+1}, \text{ if } \omega = 2m \\ -1 + F_{2m+2} \cdot L_{2m}, \text{ if } \omega = 2m+1 \end{cases}$$

where $F_\omega \sim L_\omega$, $F_{2m-1} \sim L_{2m+1}$ and $F_{2m+2} \sim L_{2m}$.

(2) If ω is odd, then by applying [26, page 96, Formula 13] we get

$$L_{\omega+1} \cdot L_{\omega-1} = -1 + L_\omega^2.$$

If ω is even, then by [26, page 112, Theorem 7.5] we have

$$L_{\omega+1} \cdot L_{\omega-1} = -5 + L_\omega^2.$$

(3) Applying Koshy [25, page 148-149, Formulas 28,48], we also have

$$P_{2\omega} = 2P_\omega \cdot Q_\omega = \pm 2 + 2P_{\omega+1} \cdot Q_{\omega-1}$$

where $P_\omega \sim Q_\omega$ and $P_{\omega+1} \sim Q_{\omega-1}$.

Let m be unlimited. The following example produces unlimited positive integers of the form $n = d_0d_1...d_m$ (here d_i is limited for every $i \geq 1$ limited) which can be written in the form $s + \omega_1\omega_2$, where $\omega_1 \sim \omega_2$.

Example 10.10 Let n be an unlimited positive integer and for $1 \leq i \leq n-1$, set $d_i = 2^{2^i} + 1$ $(1 \leq i \leq n-1)$. We can easily prove that

$$3 \cdot 5 \cdot 17 \cdots \left(2^{2^{n-2}} + 1\right)\left(2^{2^{n-1}} + 1\right) = \left(2^{2^{n-1}} - 1\right)\left(2^{2^{n-1}} + 1\right) = -1 + 2^{2^{n-1}} \cdot 2^{2^{n-1}},$$

which is of the form $\omega_1\omega_2$ and $-1 + \varpi_1\varpi_2$ where $\omega_1 \sim \omega_2$ and $\varpi_1 \sim \varpi_2$.

11 Is there an unlimited prime number of the form (A_2) or (R_2) or (AR_2)?

In this section, we mention some conjectures in number theory (which are considered to be classic open problems in mathematics) that are similar to the one about unlimited primes of the form $s+\omega_1\omega_2$ where ω_1 and ω_2 have the same order.

In [22], Iwaniec proved that there are infinitely many n such that $1+n^2$ is either prime or the product of two primes. From this result, we deduce that there are unlimited positive integers ω such that $1+\omega^2$ is either prime or the product of two primes. Thus, by Proposition 8.3 (for $b=0$), there exists an unlimited prime number of the form (AR_2) or there exist two prime numbers p, q with $q \cong \infty$ such that $p \cdot q$ is of the form (AR_2). More precisely, by [22] and Proposition 8.3, there exists an unlimited prime number p such that

$$p = \begin{cases} 2+(n-1)(n+1), & \text{for } n \text{ even} \\ 5+(n-2)(n+2), & \text{otherwise} \end{cases}$$

or there exist two prime numbers p, q with $q \cong \infty$ such that

$$p \cdot q = \begin{cases} 2+(n-1)(n+1), & \text{for } n \text{ even} \\ 5+(n-2)(n+2), & \text{otherwise} \end{cases}$$

where $n \cong \infty$. Similarly, if Conjectures 9.1 and 9.2 are true, then by Proposition 8.3 there are infinitely many primes of the form (AR_2).

Next, according to the theorems that have been proven that there are infinitely many prime numbers of the form $s+w_1w_2$ where $s \in \mathbb{Z}$ is small and $w_1, w_2 \in \mathbb{N}$ are sufficiently large, we can prove the following proposition:

Proposition 11.1 *There are infinitely many primes of the form* $s+\omega_1\omega_2$ *where* $\omega_1, \omega_2 \in \mathbb{N}$ *are unlimited with* $\gcd(\omega_1, \omega_2) = 1$.

Proof Let ω be unlimited. By Dirichlet's Theorem (see, for example, Nathanson [28, Theorem 10.9, page 347]) there are infinitely many primes of the form $1+2^{\omega} \cdot t$, where $t \cong +\infty$. When t is divisible by an unlimited odd prime power, it is done. Otherwise, we apply Proposition 6.1 and Theorem 4.5 when t is either limited or the product of a limited integer t' and an unlimited prime power of the form 2^x. The proof is finished. □

From the above proposition, it is not possible to deduce that there are infinitely many primes of the form (A_2). However, if there are infinitely many prime numbers of the form $a + n^2$ where $a \in \mathbb{Z}^*$ is limited, then by Proposition 8.3 (for $b = 0$) there are infinitely many prime numbers of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In fact, by Proposition 8.3 we get:

$$a + n^2 = \begin{cases} a + 1 + (n-1)(n+1), & \text{if } n \text{ is even} \\ a + 4 + (n-2)(n+2), & \text{otherwise.} \end{cases}$$

There are open problems regarding the digits of unlimited integers, whose validity implies the existence of unlimited prime numbers of the form (AR_2). In De Koninck and Mercier [10, Problem 260, page 159], we do not know whether there are infiniely many prime numbers whose digits (in the decimal expansion) equal 1. In fact, if there exists an unlimited positive integer ω such that $p = \underbrace{11\ldots1}_{\omega\text{-times}}$ is prime[5], then

$$p = \begin{cases} 1 + (1\underbrace{0\ldots0}_{(k-1)\text{-times}}1)(\frac{10^{2k}-1}{1\underbrace{0\ldots0}_{(k-1)\text{-times}}1} + \underbrace{11\ldots1}_{k\text{-times}}), \text{ if } \omega = 2k+1 \\ 1\underbrace{0\ldots0}_{(k-1)\text{-times}}1 \times \underbrace{11\ldots1}_{k\text{-times}}, \text{ if } \omega = 2k. \end{cases}$$

In both cases, p can be written as in (A_2). Applying Theorem 4.5, p is of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}$ and $\gcd(\omega_1, \omega_2) = 1$. Similarly, we do not know whether there are infinitely many prime numbers p of the form 100...0d where $1 \leq d \leq 9$. If this fact is true, then there are unlimited prime numbers of the form $s + \omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$. In fact, if $p = 1\underbrace{00\ldots0}_{\omega\text{-times}}d$ for some $\omega \cong \infty$, then $p = d + 10^\omega$ and from Example 1.2, p is of the form (AR_2).

Definition 11.2 Bunyakovsky's conjecture [8] states that under special conditions, polynomial integer functions of degree m greater than one generate infinitely many primes. These conditions are: (i) the coefficients of the polynomial have to satisfy: gcd(coefficients) $= 1$, (ii) the polynomial has to be irreducible, that is to say, not divisible by any other polynomial of degree d with $0 \leq d < m$. We deduce from this conjecture that if $f(x)$ is an irreducible polynomial with integer coefficients and if N denotes the greatest common divisor of the numbers $f(x)$, x running over all integers, then the polynomial $f(x)/N$ takes prime number values for infinitely many x. Under

[5] The first positive integers n such that $\underbrace{11\ldots1}_{n\text{-times}}$ is prime are 2, 19, 23, 317, 1031, ...

this conjecture, we can also prove that there are infinitely many primes of the form (AR_2).

Proposition 11.3 *Assuming Bouniakowsky conjecture [8], there are infinitely many primes of the form* $2+\omega_1\omega_2$ *where* $\omega_1,\omega_2 \in \mathbb{N}$ *are unlimited with* $\omega_1 \sim \omega_2$ *and* $\gcd(\omega_1,\omega_2)=1$.

Proof Consider the polynomial $p(x)=x^2+x+4$ which is irreducible, and for all integers x the numbers $p(x)$ are even. Since $f(0)=4$ and $f(1)=6$, we deduce that for x running over all integers the greatest common divisor of the numbers $p(x)$ is 2. Consequently, it follows from the conjecture of Bouniakowsky that for infinitely many integers x the number $p(x)/2$ is prime, say p. Thus, $p=2+x(x+1)/2$ which is of the form $2+\omega_1\omega_2$ where $\omega_1,\omega_2\in\mathbb{N}$ are unlimited with $\omega_1\sim\omega_2$. As $x(x+1)/2$ is always an integer and $\gcd(\omega_1,\omega_2)=1$, the proof is finished. □

Theorem 11.4 *The following statements are equivalent:*

(**S**) *There exist unlimited positive integers* ω_1,ω_2 *such that* $\omega_1\sim\omega_2$ *and a limited* $s\in\mathbb{Z}$ *such that* $s+\omega_1\cdot\omega_2$ *is prime.*

(**C**) *There exist* $k\geq 1$ *and* $s\in\mathbb{Z}$ *such that for every* n *there exist* m,x *such that* $x>n,m$ *and the value of the form* $k\cdot x^2+m\cdot x+s$ *is a prime number.*

Proof Assume (**S**) holds. Without loss of generality, assume that $\omega_1\leq\omega_2$. Then $\omega_2 = k\cdot\omega_1+\mu$ where $k\geq 1$ is limited and $\mu<\omega_1$ and $s+\omega_1\cdot\omega_2 = k\cdot\omega_1^2+\mu\cdot\omega_1+s$. Thus we have (take $x=\omega_1, m=\mu$)

$$\exists x\forall^{st}n\,(x>n\wedge\exists m\,(m<x\wedge k\cdot x^2+m\cdot x+s \text{ is prime})).$$

Using Idealization we rewrite this statement as

$$\forall^{st}n\exists x\,(x>n\wedge\exists m\,(m<x\wedge k\cdot x^2+m\cdot x+s \text{ is prime})).$$

By Transfer (**C**) holds.

Conversely, if (**C**) holds, then by Transfer there exist standard $k>0$ and $s\in\mathbb{Z}$ such that

$$\forall n\exists x\,(x>n\wedge\exists m\,(m<x\wedge k\cdot x^2+m\cdot x+s \text{ is prime})).$$

We take n unlimited and $\omega=x>n$ (hence, ω is unlimited) such that for some $m<\omega$ the value of $k\cdot\omega^2+m\cdot\,\omega\,+s$ is a prime number. It remains to let $\omega_1=\omega$ and $\omega_2 = k\cdot\omega\,+m$; clearly $\omega_1\sim\omega_2$ and (**S**) holds. □

An equivalent formulation of (**C**) is (**C′**): There exist $k\geq 1$ and $s\in\mathbb{Z}$ such that the value of the polynomial $k\cdot x^2+x\cdot y+s$ is a prime number for infinitely many pairs x,y with $x>y$.

12 Open questions

As our final conclusion, we propose for further research the following interesting problems:

(1) Does there exist an unlimited prime number of the form $s+\omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$?

(2) Assume that $n = s+\omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$. Is n of the form $s' + \varpi_1\varpi_2$ where $s' \in \mathbb{Z}$ is limited and $\varpi_1, \varpi_2 \in \mathbb{N}$ are unlimited with $\varpi_1 \sim \varpi_2$ and $\gcd(\varpi_1, \varpi_2) = 1$?

(3) Let n, q be unlimited with q prime. Using the ideas outlined in Proposition 10.1, is $2^n \cdot q$ of the form $s+\omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$?

(4) Let n be unlimited. Is $2^n + n$ of the form $s+\omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$?

(5) Let p, q be two unlimited primes such that $p \sim q$. Is pq of the form $s+\omega_1\omega_2$ where, $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$?

(6) Let $n = pq$, where p, q are two unlimited primes with $p \nsim q$. Is n of the form $s+\omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$?

(7) Let p_i be the i-th prime number. In the case when n is unlimited, we ask if the number $\omega_n = 2 \cdot 3 ... p_{n-1} p_n$ can be represented as $s+\omega_1\omega_2$ where $s \in \mathbb{Z}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1 \sim \omega_2$.

(8) Let ω_1, ω_2 be two unlimited positive integers such that $\omega_1 \sim \omega_2$. Is $\omega_1\omega_2$ of the form $s+\varpi_1\varpi_2$ where $s \in \mathbb{Z}^*_-$ is limited and $\varpi_1, \varpi_2 \in \mathbb{N}$ are unlimited with $\varpi_1 \sim \varpi_2$?

(9) Let $k \geq 2$ be limited. As a generalization of Theorem 9.5, are there unlimited positive integers n such that $n^{2k+1} = s^{2k+1} + \omega^2$ where $s \in \mathbb{N}^*$ is limited and $\omega \in \mathbb{N}$ has only unlimited prime factors?

(10) Are there unlimited positive integers n such that $n^3 = s + \omega^2$ where $s \in \mathbb{N}^*$ is limited and $\omega \in \mathbb{N}$ has only unlimited prime factors?

(11) Are there unlimited positive integers n such that $n^3 = s+\omega_1\omega_2$ where $s \in \mathbb{N}^*$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ with $\omega_1 \neq \omega_2$ and $\omega_1 \sim \omega_2$?

(12) Let $S(t)$ denote the sum of the digits of t. Does there exist an unlimited prime number of the form $s+\omega_1\omega_2$ where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $S(\omega_1), S(\omega_2)$ limited?

(13) Let n be an unlimited positive integer, and let F_n be the Fibonacci number. We ask whether F_n can be represented in the form $F_n = s+\omega_1\omega_2\omega_3$ where $s \in \mathbb{Z}$ is limited and $\omega_1, \omega_2, \omega_3 \in \mathbb{N}$ are unlimited with $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 3$.

(14) Is every unlimited positive integer n of the form $n = \omega_1 + \omega_2 + \omega_3\omega_4$ where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \omega_j$ and $\gcd(\omega_i, \omega_j) = 1$ for $i \neq j$?

Acknowledgements. It gives us a great pleasure to thank Professor Karel Hrbáček for his help to prove Lemma 7.5 and Theorem 11.4. The authors would also like to thank the referees whose comments helped us to improve the presentation of the paper.

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University 08 Mai 1945 Guelma, Department of Mathematics, 24000 Guelma, Algeria.

Laboratory of Pure and Applied Mathematics (LMPA), University of M'sila, B.P. 166, Ichbilia, 28000 M'sila, Algeria.

bellaouar.djamel@univ-guelma.dz, abdelmadjid.boudaoud@univ-msila.dz

Received: 21 November 2022 Revised: 21 July 2025