Journal of Logic & Analysis 16:5c (2024) 1–3
ISSN 1759-9008

Errata: multi-level nonstandard analysis and the axiom of choice

KAREL HRBACEK

The formulation of Proposition 3.3 (Factoring Lemma) on page 10 of the following paper is incorrect:

Multi-level nonstandard analysis and the axiom of choice, J. Logic & Analysis 16 (2024) 1–29

Consequently, the notion of admissible formula (Definition 4.1 on page 11) has to be weakened. While the new formulations suffice for the proofs in Sections 5 and 6, Proposition 4.5 on page 13 cannot be proved in **SCOTS** and Section 7, which depends on this proposition in an essential way, is unjustified.

The claim that **SPOTS** is conservative over $\mathbf{ZF} + \mathbf{ACC}$ made in Theorem 4.7 (and referred to in the Abstract and the penultimate sentence of Section 1) is also unproved. The argument given for it in Subsection 8.5 fails because in the absence of **ADC** the forcing with $\mathbb{P}$ might add new countable sets. This invalidates the Factoring Lemma and hence the claim that **HO** holds in $\mathcal{M}_\infty$ made in the paragraph before Proposition 8.6.

The results in the rest of the paper are correct, modulo some modifications that are listed below. A few other small mistakes and typos are also corrected.

(**1**) To correct the Factoring Lemma, the last three paragraphs of Section 3 (page 10) should be replaced by the following text.

We fix $r \in \mathbb{N}$. For $f \in \mathbb{V}^{I^{r+n}}$ and $\mathbf{i} \in I^r$ we define $f_{\mathbf{i}} \in \mathbb{V}^{I^n}$ by $f_{\mathbf{i}}(\mathbf{j}) = f(\mathbf{i},\mathbf{j})$ for all $\mathbf{j} \in I^n$ and let $\Omega_n\left([f]_{r+n}\right) = [F]_r$ where $F(\mathbf{i}) = [f_{\mathbf{i}}]_n$ for all $\mathbf{i} \in I^r$. It is routine to check that $\Omega_n : \mathbb{V}_{r+n} \to (\mathbb{V}_n)^{I^r}/U_r$ is well-defined and it is an isomorphism of the structures $(\mathbb{V}_{r+n}, \in_{r+n})$ and $(\mathbb{V}_n, \in_n)^{I^r}/U_r$.

We use the notations $r \oplus \mathsf{a} = \{r + s \mid s \in \mathsf{a}\}$ and $r \boxplus \mathsf{a} = r \cup (r \oplus \mathsf{a})$. Note that if $\mathsf{a} = n = \{0, \ldots, n-1\} \in \mathbb{N}$, then $r \boxplus n = r + n$.

Proposition 3.3 (Factoring Lemma) *The mapping Ω_n is an isomorphism of the structures*

$$(\mathbb{V}_n, \in_n, \mathbb{V}_{\mathsf{a}}, \Pi_{\mathsf{a}}^{\mathsf{b}}\,;\ \mathsf{a},\mathsf{b} \subset n, |\mathsf{a}| = |\mathsf{b}|)^{I^r}/U_r$$

and

$$(\mathbb{V}_{r+n},\ \in_{r+n},\ \mathbb{V}_{r\boxplus\mathsf{a}},\ \Pi_{r\boxplus\mathsf{a}}^{r\boxplus\mathsf{b}}\,;\ \mathsf{a},\mathsf{b} \subset n, |\mathsf{a}| = |\mathsf{b}|).$$

Published: August 2024 https://doi.org/10.4115/jla.2024.16.5c

(**2**) Definition 4.1 (page 11) and the paragraph that follows it have to be modified.

A formula Φ is *admissible* if labels appear in it only as subscripts and superscripts of $\mathbb{S}$ and $\mathbf{I}$, and all quantifiers are of the form $\forall v \in \mathbb{S}_{\mathsf{a}}$ and $\exists v \in \mathbb{S}_{\mathsf{b}}$.

Definition 4.1 (*Admissible formulas*)

- $u = v$, $u \in v$, $v \in \mathbb{S}_{\mathsf{a}}$ and $\mathbf{I}_{\mathsf{a}}^{\mathsf{b}}(u) = v$ are admissible formulas.
- If Φ and Ψ are admissible, then $\neg\Phi$, $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \to \Psi$ and $\Phi \leftrightarrow \Psi$ are admissible.
- If Φ is admissible, then $\forall v \in \mathbb{S}_{\mathsf{a}}\, \Phi$ and $\exists v \in \mathbb{S}_{\mathsf{b}}\, \Phi$ are admissible.

Let r be a variable that ranges over standard natural numbers and does not occur in the formula Φ. The formula $\Phi^{\uparrow r}$ is obtained from Φ by replacing each occurrence of $\mathbb{S}_{\mathsf{a}}$ with $\mathbb{S}_{r \boxplus \mathsf{a}}$ and each occurrence of $\mathbf{I}_{\mathsf{a}}^{\mathsf{b}}$ with $\mathbf{I}_{r \boxplus \mathsf{a}}^{r \boxplus \mathsf{b}}$. In particular, if Φ is a formula where only the symbols $\mathbb{S}_{\mathsf{n}}$ for $\mathsf{n} \in \mathbb{N} \cap \mathbb{S}_0$ occur, then $\Phi^{\uparrow r}$ is obtained from Φ by shifting all levels by r.

(**3**) The proof of Proposition 4.2 (page 12) also needs to be modified.

Proof The axiom (3) in **IS** follows from Proposition 3.2; the rest is obvious.

To prove that **GT** holds, we recall that the mapping $\Pi_{\mathsf{a}}^{\infty} : \mathbb{V}_{\mathsf{a}} \to \mathbb{V}_{\infty}$ is an elementary embedding, ie, if $\psi(v_1, \ldots, v_\ell)$ is any $\in$–formula and $x_1, \ldots, x_\ell \in \mathbb{S}_{\mathsf{a}}$, then $\psi^{\mathbb{S}_{\mathsf{a}}}(x_1, \ldots, x_\ell) \leftrightarrow \psi(x_1, \ldots, x_\ell)$. Using this observation twice shows that for all $x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{a}}$:

$$\forall x \in \mathbb{S}_{\mathsf{a}}\ \phi(x, x_1, \ldots, x_k) \to \forall x \in \mathbb{S}_{\mathsf{a}}\ \phi^{\mathbb{S}_{\mathsf{a}}}(x, x_1, \ldots, x_k) \to \forall x\ \phi(x, x_1, \ldots, x_k)$$

HO is justified by the Factoring Lemma (take n so that all labels occurring in Φ are proper subsets of n) and Łoś's Theorem (specifically, by the fact that for each n the canonical embedding of $(\mathbb{V}_n, \in_n, \mathbb{V}_{\mathsf{a}}, \Pi_{\mathsf{a}}^{\mathsf{b}})$ into its ultrapower by U_r is elementary). $\square$

(**4**) Proposition 4.5 is wrong. It is replaced by the following proposition, which should precede Proposition 4.4, in the proof of which it is used.

Proposition Let $\Psi(v, v_1, \ldots, v_k)$ be an admissible formula and let $\Phi(v_1, \ldots, v_k)$ be either $\forall v\, \Psi(v, v_1, \ldots, v_k)$ or $\exists v\, \Psi(v, v_1, \ldots, v_k)$. Then **HO** holds for Φ.

Proof It suffices to prove the existential version. Fix $r \in \mathbb{N} \cap \mathbb{S}_0$, $\mathsf{a} \in \mathcal{P}^{\mathbf{fin}}(\mathbb{N}) \cap \mathbb{S}_0$ and $x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{a}}$. We have $\exists x\, \Psi(x, x_1, \ldots, x_k)$ if and only if there exists n such that $\exists x \in \mathbb{S}_{\mathsf{n}}\, \Psi(x, x_1, \ldots, x_k)$ if and only if (by applying **HO** to this admissible formula) there exists n such that $\exists x \in \mathbb{S}_{\mathsf{n}+r}\, \Psi^{\uparrow r}(x, \mathbf{I}_{\mathsf{a}}^{r \oplus \mathsf{a}}(x_1), \ldots, \mathbf{I}_{\mathsf{a}}^{r \oplus \mathsf{a}}(x_k))$ if and only if $\exists x\, \Psi^{\uparrow r}(x, \mathbf{I}_{\mathsf{a}}^{r \oplus \mathsf{a}}(x_1), \ldots, \mathbf{I}_{\mathsf{a}}^{r \oplus \mathsf{a}}(x_k))$. $\square$

Hence the axioms of **SPOT**, in particular **SP**, postulated in **SPOTS** only about the level $\mathbb{S}_0$, hold there about every level $\mathbb{S}_{\mathsf{n}}$.

(**5**) The formula in the definition of **SPOTS** on page 13 should be:

$$\exists \nu \in \mathbb{N} \cap \mathbb{S}_1 \; \forall^{\mathbf{st}} n \in \mathbb{N} \; (n \neq \nu).$$

(**6**) The definition of **J** in the proof of Ramsey's Theorem on page 15 should be:

$$\mathbf{J} = \mathbf{I}^{\{0,\ldots,p-1,p+1,\ldots,n\}}_{\{0,\ldots,n-1\}}$$

(**7**) Case 2 on the list of properties required of Jin's universes (Section 6, page 15) should say:

For $j' > j$, Countable Idealization holds from $\mathbb{V}_j$ to $\mathbb{V}_{j'}$: Let ϕ be an $\in$–formula with parameters from $\mathbb{V}_{j'}$. Then

$$\forall n \in \mathbb{N}_j \; \exists x \in \mathbb{V}_{j'} \; \forall m \leq n \; \phi^{\mathbb{V}_{j'}}(m, x) \;\leftrightarrow\; \exists x \in \mathbb{V}_{j'} \; \forall n \in \mathbb{N}_j \; \phi^{\mathbb{V}_{j'}}(n, x).$$

(**8**) In the proof of Proposition 6.3 on page 17, "a special **st**–$\in$–formula" should be "an $\mathbb{N}$–special formula".

(**9**) The last sentence in Section 6:

> "Their existence at higher levels in **SPOTS** follows from Proposition 4.5"

should be

> "Their existence at higher levels follows from Proposition 6.3 and the observation (in the proof of Proposition 6.1, Property 2) that $(\mathbb{S}_{j'}, \in, \mathbb{S}_j)$ satisfies **SPOT**."

Department of Mathematics, City College of CUNY, New York, NY 10031,

khrbacek@icloud.com

Received: 12 August 2024 Revised: 25 August 2024