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Errata: a density version of a theorem of Banach

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The proof of a result in the following paper contains an error:

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Fortunately, this error is straightforward to correct.

In the proof of the (i⇒ii) case of Theorem 1.2, Corollary 4 is used to select a $J \subseteq I$ with $\bar{d}(J) > 0$. However, while the upper density of J would be positive relative to I, this does not mean that it is positive as a subset of $\mathbb{N}$. To correct that, Corollary 4 needs to be strengthened.

(**1**) Start with the following reformulation/strengthening of Lemma 2.

Lemma 2 *Let $(X, \mathcal{A}, \mu)$ be a probability measure, $a > 0$, $I_0 \subseteq \mathbb{N}$ with $\bar{d}(I_0) > b > 0$, and $A_n \in \mathcal{A}$ with $\mu(A_n) \geq a$ for all $n \in I_0$. For some $I \subseteq I_0$ with $\bar{d}(I) \geq ab$, $\{A_n\}_{n \in I}$ has the finite intersection property.*

(Note that the original version of Lemma 2 is a special case of this one, with $I_0 = \mathbb{N}$.)

(**2**) Replace the first line of the proof of Lemma 2 (Section 3.1) by the following 6 lines:

Adopt the following notation. If $J \subseteq \mathbb{N}$ and $N \in \mathbb{N}$ write $J \wedge N = J \cap \{1, \ldots, N\}$ and $|J \wedge N| =$ the (finite) cardinality of $J \wedge N$.

Given I_0 with $\bar{d}(I_0) > b$ let η_n be an increasing sequence of natural numbers with $|I_0 \wedge \eta_n|/\eta_n > b$ for all n. For $x \in X$ and $n \in \mathbb{N}$ define:

$$F_n = \frac{1}{|I_0 \wedge \eta_n|} \sum_{k \in I_0 \wedge \eta_n} \chi_{A_n}(x)$$

There are two cases:

(**3**) Replace the proof of Case 1 (bottom of page 4) with the following:

Case 1: $\limsup_n F_n(x) \geq a$ for some x. Put $I = \{n \in I_0 : x \in A_n\}$, then $\{A_n\}_{n \in I}$ has the finite intersection property. Observe:

$$\frac{|I \wedge \eta_n|}{\eta_n} = \left(\frac{|I \wedge \eta_n|}{|I_0 \wedge \eta_n|}\right)\left(\frac{|I_0 \wedge \eta_n|}{\eta_n}\right) > F_n(x)b$$

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so if we let $n \to \infty$ along a subsequence n_k witnessing $\limsup_n F_n(x) \geq a$, we get $\bar{d}(I) \geq ab$.

(**3**) Replace the displayed calculation on page 2 from the proof of Case 2 by the following calculation. (The proof of this case is otherwise unchanged.)

$$\begin{aligned} a &\leq \frac{1}{|I_0 \wedge \eta_N|} \sum_{k \in I_0 \wedge \eta_N} \int \chi_{A_k}(x) = \int_X F_N d\mu \\ &= \int_{B \cap C} F_N d\mu + \int_{B \setminus C} F_N d\mu + \int_{X \setminus B} F_N d\mu \\ &\leq r\mu(B \cap C) + a\mu(B \setminus C) + 1\mu(X \setminus B) \end{aligned}$$

(**4**) The corollaries now have the following stronger statements:

Corollary 3 *Let $(X, \mathcal{A}, \mu)$ be a probability measure, $a > 0$, $I_0 \subseteq \mathbb{N}$ with $\bar{d}(I_0) > b > 0$ and $A_n \in \mathcal{A}$ with $\mu(A_n) \geq a$ for all $n \in I_0$. For some $I \subseteq I_0$ with $\bar{d}(I) \geq ab$ and every finite $J \subseteq I$, $\mu(\bigcap_{n \in J} A_n) > 0$.*

Corollary 4 *Let $(X, \mathcal{A}, \mu)$ be a* finitely additive *probability measure, $a > 0$, $I_0 \subseteq \mathbb{N}$ with $\bar{d}(I_0) > b > 0$, and $A_n \in \mathcal{A}$ with $\mu(A_n) \geq a$ for all $n \in I_0$. For some $I \subseteq I_0$ with $\bar{d}(I) \geq ab$ and every finite $J \subseteq I$, $\mu(\bigcap_{n \in J} A_n) > 0$.*

(**5**) In the proof of Corollary 3, "density at least a" should be replaced by "density at least ab". Likewise, In the proof of Corollary 4, "$\bar{d}(I) \geq a$" should be replaced by "$\bar{d}(I) \geq ab$".

(**6**) Finally, the proof of the main result needs one emendation. The fourth line from the end (before the references) should now be:

By Corollary 4, there is a subset $J = \{n_m\}_m \subseteq I$ such that $\bar{d}(J) > \alpha^2 r/M$ and such that for every $N \in \mathbb{N}$, $\mu\big(\bigcap_{m=1}^{N} A_{n_m}\big) > 0$.

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