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Effective bases and notions of effective second countability in computable analysis

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Abstract: We compare the approaches of Schröder and Weihrauch to computable topology. These approaches are both based on the theory of representations, but they focus on a priori different representations of open sets. We show that the two approaches are compatible. While it is well known that Schröder's approach is more general because it can be applied to non-second countable spaces, we insist on the fact that it is also more general even when dealing only with second countable spaces, because a represented space can be second countable without being *effectively second countable*.

We revisit Schröder's Effective Metrization Theorem, by showing that it characterizes those represented spaces that embed into computable metric spaces: those are the computably second countable strongly computably regular represented spaces.

Finally, we study different forms of open choice problems. We show that having a computable open choice is equivalent to being computably separable, but that the "non-total open choice problem", i.e., open choice restricted to open sets that have non-empty complement, interacts with effective second countability in a satisfying way.

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1 Introduction

Choosing a good framework and correct definitions is one of the most important and sometimes most challenging problems in computable mathematics. Indeed, it is often the case that once a good framework has been found, many results that seemed non-trivial before can easily be proven. To see that this is the case in computable analysis, one can for instance look at early articles of Markov [43] and Lacombe [42], which translated to today's vocabulary end up having very little content, and compare them to Pauly's 2016 survey paper [49] which deals with a great many notions and results in a mere 22 pages.

We are concerned here with computable topology as studied in Weihrauch's German school of computable analysis, we follow a "Type 2 approach": the basic notion of computable function is set

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on Baire space, and it is transferred from Baire space to other spaces with cardinality that of the continuum thanks to partial surjections which are called *representations*.

Our starting point is that within Weihrauch's school of computable analysis two approaches to studying computable topology coexist.

Namely:

- The approach devised by Matthias Schröder in his PhD thesis [54] (see also [53, 55]), which uses the representation of the Sierpiński space to define a representation of open sets on any represented space equipped with the final topology of its representation.
- The approach that Weihrauch developed with Grubba in [66], which relies on a numbered base $(B_i)_{i\in\mathbb{N}}$ for a space X to define a representation of X and a representation of the open sets of X.

Ideas very similar to those of Schröder were developed independently by Paul Taylor [60] and Martín Escardó [15], outside the framework of represented sets.

It is clear that Schröder's approach is more general than Weihrauch's, in that it applies to spaces that need not be second countable. Yet the exact relationship between these two approaches had not been clarified until now.

In the present paper, we explain the following:

- The approaches of Schröder and Weihrauch to computable topologies are compatible.
- The approach of Schröder is strictly more general than Weihrauch's, *even for second countable spaces*: not only does it encompass more topological spaces, but even on the topological spaces that fall under both approaches, purely computability theoretical phenomena occur in Schröder's approach that are not accounted for in Weihrauch's approach.
- Weihrauch and Grubba's notion of "computable topological space" in fact corresponds to a form of *effective second countability*.
- This notion of effective second countability is often the most useful one, yet other weaker notions can be relevant, because the "effectivization" of some theorems can already be obtained thanks to those weaker notions.

Another way to understand the present article is as follows.

One of the key notions to study topology thanks to the theory of represented spaces is that of *admissible representation*, because continuous maps between admissibly represented spaces always have a continuous realizer. Also, when studying computable topology, *computably admissible representations* play a central role, because these are the representations that guarantee the equivalence between "being computable" and "being effectively continuous". See Section 2 for more details.

One of the earliest representations ever studied is the *standard representation* introduced by Kreitz and Weihrauch [39]. Given a second countable T_0 space X equipped with a countable subbase $(B_i)_{i\in\mathbb{N}}$, this representation is given by

$$\rho(f) = x \iff \mathrm{Im}(f) - 1 = \{n \in \mathbb{N},\ x \in B_n\},$$

where $\mathrm{Im}(f) - 1 = \{n \in \mathbb{N},\ \exists k \in \mathbb{N}, n+1 = f(k)\}$. Note that every totally numbered subbase of X induces a standard representation, and thus each second countable space admits multiple standard representations. However, Kreitz and Weihrauch have shown that, up to continuous equivalence of representations, the choice of a standard representation is inconsequential:

Theorem 1.1 ([39]) *All standard representations of a second countable space are continuously equivalent. Every admissible representation of a second countable space is continuously equivalent to a standard representation.*

It is clear that not all standard representations of a second countable space need to be computably equivalent. What we want to emphasize here is that the effective analogue of the second point in the above theorem also fails: *a computably admissible representation of a second countable space does not have to be computably equivalent to a standard representation.*

Thus, while fixing a standard representation on a second countable space can be seen as a "neutral operation" from the point of view of topology, this fact does not carry over to the study of computability: in a context where representations are considered up to computable translations, studying only spaces that are equipped with standard representations amounts, implicitly, to imposing certain computability-theoretic assumptions. The representations that are computably equivalent to a standard representation are exactly those that give rise to what we call computably second countable spaces.

In this paper, we show that the notion of a computably second countable represented space is extremely robust, as it emerges from a wide variety of different approaches.

But we also show that among representations that are not computably equivalent to a standard representation, there is a range of weaker notions of effective second countability that can be, in different contexts, relevant.

1.1 Notions of effective bases and their relations

In Section 3, we introduce in detail the notions of bases that we consider. Here we only sketch the definitions.

- **Semi-effective bases.** A semi-effective base is a set of uniformly open sets that form a base. These sets are "constructively open", but the assumption that they form a base is purely classical.

- **Lacombe bases.** This is the notion of base that follows from the classical statement "a set $\mathfrak{B}$ forms a base for a topological space X if the open sets are exactly the sets that can be written as unions of elements of $\mathfrak{B}$". The effective version of this statement will say that "open sets can uniformly be written as overt unions of basic sets". This notion is named after Lacombe following the article [41]. This approach has been used by Hoyrup and Rojas [23], Amir and Hoyrup [23], Korovina and Kudinov [37, 38], Grubba and Weihrauch [17], Grubba and Weihrauch and Yu [18], and Hoyrup, Rojas, Selivanov, and Stull [24], in the second countable case, and Bauer and Lešnik [4] in a more general setting.
- **Nogina bases.** This is the notion of base that follows from the classical statement "a set $\mathfrak{B}$ forms a base for a topological space X if a set O is open if and only if for any $x \in O$ there is $B \in \mathfrak{B}$ with $x \in B \subseteq O$". Nogina was the first to define a notion a base by using an effective version of this statement [46, 47]. Recent use of it can be found in Gregoriades, Kispéter, and Pauly [16].
- **Representation subbases.** A representation subbase of a set X is a subbase which is used to define a representation of X, by saying that the name of a point should encode the characteristic function of the set of basic sets to which it belongs. Here, characteristic functions are considered to have the Sierpiński space as their codomain, rather than the discrete two-point space $\{0,1\}$. This is one of two possible generalizations of the standard representation considered by Weihrauch and Kreitz [39]. It appears for instance in Bauer [3].
- **Enumeration subbase.** This is a second definition where a subbase is used to define a representation. Here the name of a point is a countable list of names of basic sets that define a *formal neighborhood base* of this point. This notion of base applies only to second countable spaces, but possibly to uncountable bases of second countable spaces. This definition originates in [62], where Weihrauch defined representations by describing points thanks to neighborhood bases. Following Spreen [59], it was explained in Rauzy [51] how relying on a notion of formal inclusion relation and the induced notion of formal neighborhood base yields a more robust definition.

To each notion of computable base, we associate a notion of computable second countability, following the following definition scheme:

Definition 1.2 A represented space (X,ρ) is *[Nogina, Lacombe, etc] second countable* if it admits a totally numbered [Nogina, Lacombe, etc] base, i.e., if there exists a sequence $(B_i)_{i\in\mathbb{N}}$ which forms a [Nogina, Lacombe, etc] base of (X,ρ).

Note that we allow for some non-admissible and non-computably admissible representations. Indeed, some notions of computable bases automatically imply effective admissibility of the representation under scrutiny, while others do not, we can distinguish between them only because we are not

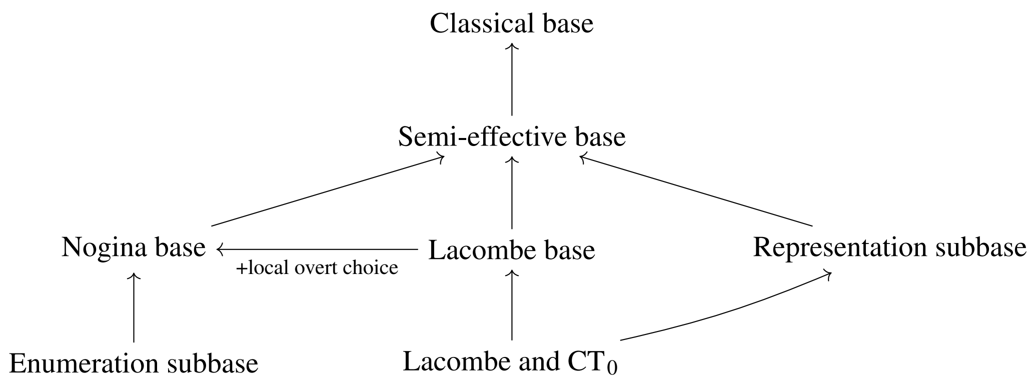

Figure 1.1: Notions of bases

restricting our attention to computably admissible representations in the first place. We abbreviate "computably admissible" by CT_0.

Our main theorem is summarized in Figures 1.1 and 1.2. Because Figure 1.1 expresses, in a concise way, 8 implications, and Figure 1.2 13 implications and a "conjectured non-implication", it seems reasonable to leave the figures as a statement of our theorem, instead of listing all these implications explicitly.

Theorem A All the implications between the different notions of computable bases and of computable second countability appear in Figure 1.1 and 1.2.

Note that the two figures express something different: when comparing notions of bases (Figure 1.1), we ask "is it the case that any base of this type is also automatically a base in that sense". (In the case of subbases we allow ourselves to replace a subbase by the base it induces.) On the other hand, when comparing notions of second countability (Figure 1.2), the question we consider is: "must a space that admits a base in this sense also admit a base in that sense". Those bases can, a priori, be very different from one another.

The five equivalent notions that appear at the bottom of Figure 1.2 define what we call *computable second countability*. Note that three of the characterizations we present were obtained independently by Neumann, Pauly, Pradic, and Valenti [45].

If (X, ρ) is a represented space, and Y is a subset of X, there are two natural representations of the open sets of Y: one can first restrict ρ to Y, and then take the associated Sierpiński representation, or first take the Sierpiński representation for X, and consider the trace of this representation on Y. If these two representations agree, Y is called *a computably sequential* subset of X (this notion was introduced by Bauer in the context of synthetic topology under the name *intrinsic subset* [3]). An important feature of computable second countability is the following:

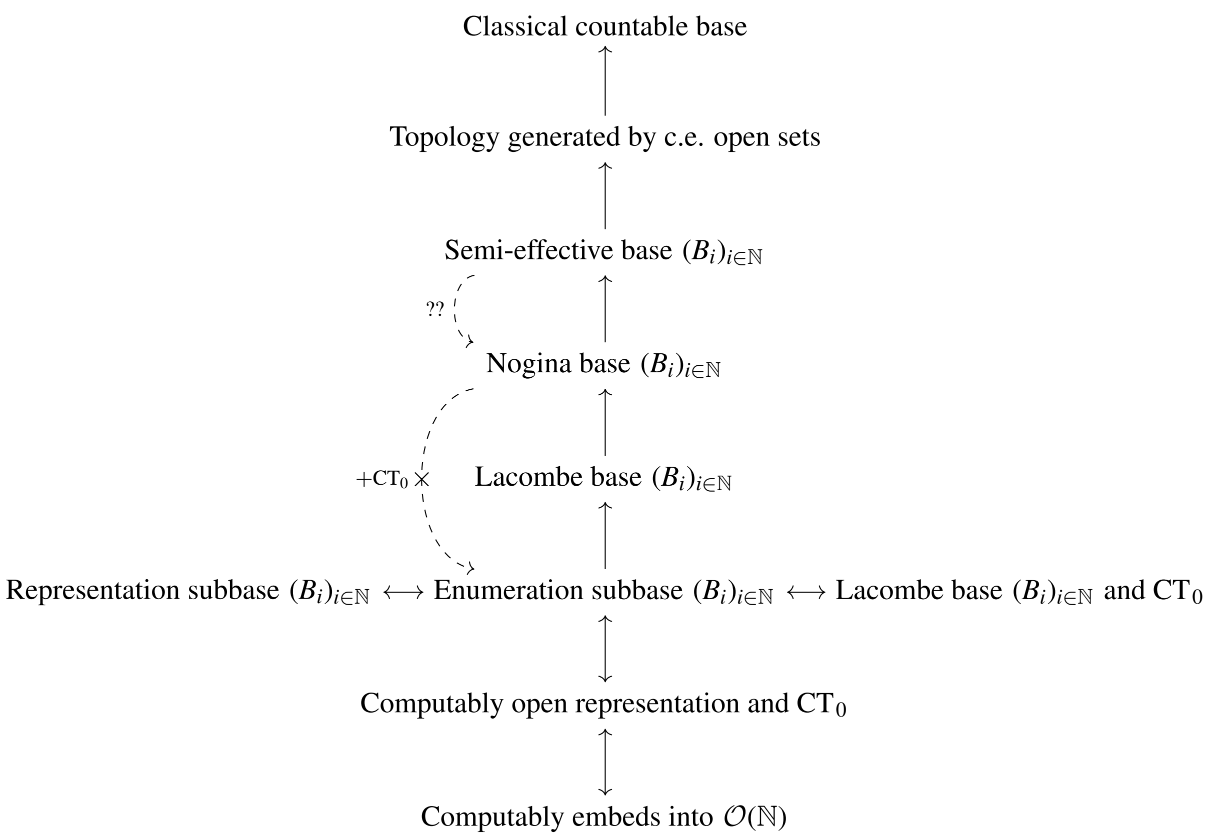

Figure 1.2: Notions of computable second countability

Theorem B Let (X, ρ) be a computably second countable represented space, let $Y \subseteq X$ be a subset of X, and equip it with the induced representation $\rho_{|Y}$. Then $(Y, \rho_{|Y})$ is also computably second countable, and it is a computably sequential subset of (X, ρ).

1.2 More on the Weihrauch–Grubba approach

Let us quote the Weihrauch–Grubba definition of a computable topological space that appears in [66]. Denote by $W_i = \mathrm{dom}(\varphi_i)$ the usual numbering of c.e. subsets of $\mathbb{N}$.

Definition 1.3 ([66]) A *computable topological space* is a pair $(X, (B_i)_{i\in\mathbb{N}})$, where X is a set and $(B_i)_{i\in\mathbb{N}}$ is the base of a T_0 topology on X for which there exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any i, j in $\mathbb{N}$:

$$B_i \cap B_j = \bigcup_{k\in W_{f(i,j)}} B_k.$$

In fact, the above definition is systematically studied together with two representations that are induced by the base $(B_i)_{i\in\mathbb{N}}$: a representation of points and a representation of open sets.

Definition 1.4 ([66]) Let $(X, (B_i)_{i\in\mathbb{N}})$ be a computable topological space as above. Define a representation $\theta^+ :\subseteq \mathbb{N}^\mathbb{N} \to \mathcal{O}(X)$ of the open sets of X by:

$$\theta^+(f) = \bigcup_{\{n,\, \exists p\in\mathbb{N}, f(p)=n+1\}} B_n.$$

Define also a representation ρ of X by the following formula:

$$\rho(f) = x \iff \mathrm{Im}(f) = \{n \in \mathbb{N},\, x \in B_n\}.$$

Thus, in the Weihrauch–Grubba approach, a numbered basis is used to define both a representation of open sets and a representation of points. This is not the case in Schröder's approach to computable topology, where the representation of open sets that is considered is always the one associated to the Sierpiński representation. A priori, this could lead to some conflicts, if the two approaches focus on non-equivalent representations of open sets, but we show that this can never arise. We summarize this in Figures 1.3, 1.4 and 1.5.

- In the Weihrauch–Grubba approach, a base is used to define two representations, as represented on Figure 1.3.
- Following Schröder's approach, the representation of points automatically induces a representation of open sets: this is represented on Figure 1.4.
- We then express the fact that the Weihrauch–Grubba and Schröder approaches are compatible via a commutative diagram represented on Figure 1.5.

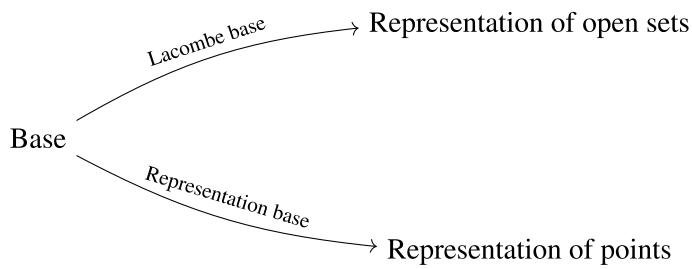

Figure 1.3: Weihrauch–Grubba approach

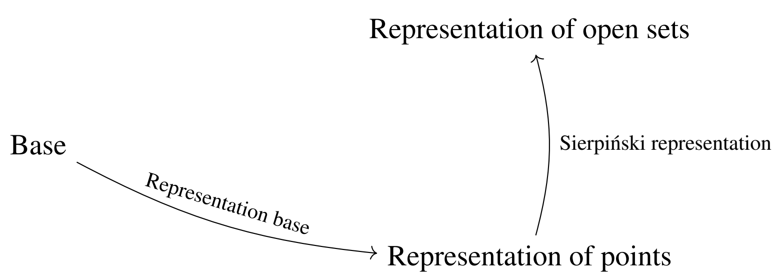

Figure 1.4: Sierpiński representation in Schröder's approach

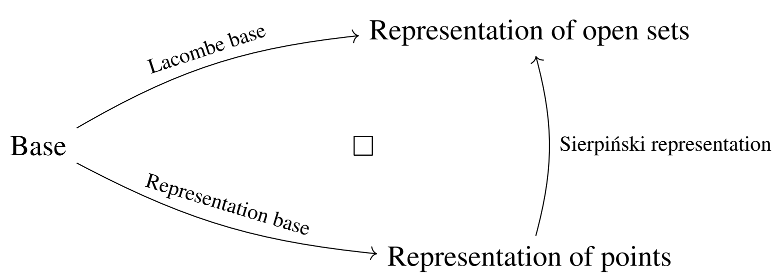

Figure 1.5: Compatibility of the approaches

Matthew de Brecht has investigated the Weihrauch–Grubba notion of computable topological space in [10]. He proved in particular that each computable function f that computes intersections for a certain base, i.e., the function f that appears in

$$B_i \cap B_j = \bigcup_{k \in W_{f(i,j)}} B_k,$$

can be associated to a unique maximal topological space which is a precomputable quasi-Polish space (de Brecht, Pauly, and Schröder [12]). This provides a method for uniquely specifying a topological space and an associated representation thanks to a finite amount of data. This approach is orthogonal to the one presented here: we show that Schröder's approach leads to the same representations as those introduced thanks to Definition 1.4, but foregoing the function f entirely, de Brecht has shown what is exactly the information that can be recovered from f.

1.3 Refining results that rely on effective second countability

Consider a classical mathematical theorem of the form:

(1) If a topological set X is second countable and A, then B.

Thanks to the different notions of effective second countability that we consider, the following problem now makes sense: which one of the effective second countability notions is the weakest sufficient notion to establish an effective version of this classical result?

Many results obtained in the Weihrauch–Grubba formalism could be revisited with these questions in mind, in particular the results of [64, 65].

In the present article, we are content with providing two examples of classical statements of the form (1) that require a strong and a weak version of "effective second countability" respectively. These two examples seem to us particularly important.

The first example is Matthias Schröder's Effective Metrization Theorem [52]. Using ideas of Amir and Hoyrup [1], we prove that this theorem is sharp, in that it can be written with an "if and only if" statement.

Theorem C (Schröder–Urysohn Effective Metrization) The following are equivalent for a represented space (X, ρ):

(1) (X, ρ) computably embeds into the Hilbert cube,
(2) (X, ρ) is computably second countable and strongly computably regular.

The above imply, but are not equivalent to:

(3) (X, ρ) is computably second countable and admits a computable metric that generates its topology.

Previous versions of the above theorem stated (2) $\implies$ (3). The most important implication is actually (2) $\implies$ (1). This implication cannot be recovered from (2) $\implies$ (3) since (3) is a strictly weaker statement.

We note that, modulo Schröder's results from [52], the proof of the above theorem is very easy. The reason why we emphasize it so is that it seemed to us very important to understand precisely the full strength of the Schröder–Urysohn Effective Metrization Theorem. Despite having been used in many different contexts, the full version of this theorem had not been published before.

Strong computable regularity was introduced by Schröder in [52] under the name "computable regularity", and later renamed by Weihrauch in [65], where other effective notions of regularity were considered. See Section 8 for a full definition.

By establishing that the Effective Metrization Theorem provides an embedding into the Hilbert cube, we show that the computable second countability hypothesis is necessary in this theorem. Another characterization of represented spaces that computably embed into the Hilbert cube is given by Kihara and Pauly [34, Theorem 7.1][1], in terms of representations computably admitting compact fibers.

The second example that we consider comes from the following classical statement:

(2) A second countable space is separable.

The natural expected effective version of the above statement uses the very weak form of effective second countability that we call semi-effective second countability:

Proposition 1.5 *Let (X, ρ) be a semi-effectively second countable represented space which is overt and has a computable open choice problem. Then (X, ρ) is effectively separable.*

However, we prove the following theorem, which implies that the above proposition is useless:

Theorem D A represented space (X, ρ) has a computable open choice problem if and only if it is computably separable.

We then introduce non-total open choice: open choice restricted to open sets that have a non-empty complement. We then prove:

Theorem E Having a computable non-total open choice does not imply effective separability.

A correct effective equivalent to statement (2) can be obtained by using semi-effective second countability and computable non-total open choice, see Proposition 9.4.

[1]This theorem is unfortunately absent from the published version of [34], namely [35].

1.4 Further notes on the literature

In Section 10, we discuss several additional approaches to computable topology

Two main points are addressed:

- The distinction between results that are about spaces up to homeomorphisms and results about spaces up to computable homeomorphisms.
- The relationship between the present work and the approach to computable topology developed by Kalantari and Welch [26, 27]. In particular, we present a proposition showing that their approach is not always compatible with that of Schröder.

1.5 Acknowledgements

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2 Preliminaries

The background we require is mostly contained in [49]. See also [55]. To make the paper self-contained, we still introduce the relevant notions.

2.1 Represented spaces

A *representation* of a set X is a partial surjection $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. If $\rho(u) = x$, then u is called a ρ*-name* of x.

A point of X is *computable* with respect to ρ if it admits at least one ρ-name which is a computable sequence.

A *multi-function* between sets X and Y is a relation $R \subseteq X \times Y$ which we treat as a function f mapping points of X to subsets of Y via the formula $f(x) = \{y \in Y \mid (x, y) \in R\}$. We put $\mathrm{dom}(f) = \{x \in X \mid f(x) \neq \emptyset\}$, and consider that a multi-function f between X and Y is *total* if $\mathrm{dom}(f) = X$, and that it is *partial* otherwise.

The fact that f is a total multi-function from X to Y is denoted via $f : X \rightrightarrows Y$, and we denote a partial multi-function by $f :\subseteq X \rightrightarrows Y$.

Definition 2.1 Let (X,ρ) and (Y,τ) be represented spaces, and let $f :\subseteq X \rightrightarrows Y$ be a partial multi-function. A *realizer of f with respect to ρ and τ*, or (ρ,τ)*-realizer*, is a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ which satisfies the following:

$$\forall u \in \mathrm{dom}(\rho) \cap \rho^{-1}(\mathrm{dom}(f)),\ \tau(F(u)) \in f(\rho(u)).$$

In words: F is a realizer of f if it maps the name of any point of X to a name of one of its images by f.

Let ρ and τ be representations of a set X. Say that ρ *translates* (resp. *continuously translates*) to τ if the identity on X has a computable (resp. continuous) (ρ,τ)-realizer.

We denote the fact that ρ translates to τ by $\rho \le \tau$. If $\rho \le \tau$ and $\tau \le \rho$, then ρ and τ are called *equivalent,* we define similarly *continuously equivalent representations*.

Definition 2.2 ([39, 54]) Let X be a topological space. A representation $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is called *admissible for X* if the following hold:

(1) The representation ρ is continuous,
(2) Every continuous representation $\tau :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ continuously translates to ρ.

Theorem 2.3 ([39, 54]) *Let (X,ρ) be a represented space and let (Y,τ) be an admissibly represented space. Let $f : X \to Y$ be a function. Then the following are equivalent:*

(1) *The function f is continuous with respect to the final topologies of ρ and τ.*
(2) *The function f is admits a continuous realizer.*

2.2 Representation of continuous functions

For two represented spaces (X,ρ) and (Y,τ), there is a representation, which we denote by $[\rho \to \tau]$, of all continuously realizable functions from (X,ρ) to (Y,τ).

One way to define this representation is as follows. Continuous functions on Baire space are exactly the functions that are computable modulo some oracle. The $[\rho \to \tau]$-name of a function is thus the code of a Type 2 machine that computes a realizer, together with the oracle that this machine requires to function. The fact that currying and evaluation are computable follows from the Type 2 Universal Turing Machine Theorem (Weihrauch [61]). For more details, see [49].

2.3 Sierpiński representation

The Sierpiński space $\mathbb{S}$ is $\{0,1\}$ with topology generated by $\{1\}$: $\{1\}$ is open but $\{0\}$ is not. Its standard representation $c_{\mathbb{S}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{S}$ is defined by

$$c_{\mathbb{S}}(0^\omega) = 0,$$
$$c_{\mathbb{S}}(u) = 1 \text{ for } u \neq 0^\omega.$$

In the present article, we always consider, on any represented space (X,ρ), that X is equipped with the final topology of ρ. Thus, O is open in X if and only if $\rho^{-1}(O)$ is open in Baire space (or more precisely in $\mathrm{dom}(\rho)$, equipped with the subset topology of Baire space).

Because the representation of Sierpiński space is admissible, we get the following equivalence for any represented space (X,ρ), where $\mathbf{1}_O$ denotes the characteristic function of O :

$$O \text{ is open in the final topology of } \rho \iff \mathbf{1}_O : X \to \mathbb{S} \text{ has a continuous } (\rho, c_{\mathbb{S}})\text{-realizer},$$
$$\iff \mathbf{1}_O \text{ has a } [\rho \to c_{\mathbb{S}}]\text{-name},$$

and thus we can see the representation $[\rho \to c_{\mathbb{S}}]$ as a representation of the final topology of ρ on X. We call this the *Sierpiński representation associated to* ρ. We denote by $\mathcal{O}(X)$ the represented set thus obtained.

The computable points of the representation $[\rho \to c_{\mathbb{S}}]$ are called the *c.e. open sets.*

The representation $[\rho \to c_{\mathbb{S}}]$ is also used to define the represented space $\mathcal{A}_{-}(X)$ of *closed subsets of* X *given by negative information:* the name of a closed set is a $[\rho \to c_{\mathbb{S}}]$-name of its complement.

2.4 Computably admissible representations as $\mathbf{CT}_0$ spaces

Notice that by definition of the Sierpiński representation associated to a representation ρ, the neighborhood map

$$\mathcal{N} : X \hookrightarrow \mathcal{O}\mathcal{O}(X)$$
$$x \mapsto \{O \mid x \in O\}$$

is always computable.

Theorem 2.4 ([54]) *A represented space* (X,ρ) *is admissible if and only if the neighborhood map* $\mathcal{N}$ *has a continuously realizable inverse.*

Contrary to the original definition, this characterization of admissibility can be effectivized. This lead Schröder to introducing the following definition:

Definition 2.5 ([54]) A representation is *computably admissible* if the neighborhood map $\mathcal{N}$ has a computable inverse.

When ρ is a computably admissible representation of X, we say that the represented space (X, ρ) is *computably Kolmogorov*, or CT_0.

This choice of terminology is based on the following fact: a topological space is T_0 when points are uniquely determined by the open sets to which they belong. A space is computably Kolmogorov when the previous equivalence holds computably: the name of a point can be translated to the name of a program that recognizes the open sets to which it belongs, and vice versa.

Theorem 2.6 ([54]) *A representation ρ of a set Y is computably admissible if and only if for every represented space X and every function $f : X \to Y$, we have*

$$f \text{ is computable} \iff f \text{ is continuous and } f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X) \text{ is computable.}$$

2.5 Overt sets

Let $\mathbf{X} = (X, \rho)$ be a represented space.

A subset A of X is called *overt* if the following map is computable:

$$\mathcal{O}(X) \to \mathbb{S},\ U \mapsto (U \cap A \neq \emptyset) \tag{3}$$

In other words there is an algorithm that recognizes names of open sets that meet the set A.

A set A is overt if and only if its closure is, and it is only the closure of A that is uniquely determined by the map given by (3).

We denote by $\mathcal{V}(\mathbf{X})$ the *represented space of closed overt subsets* of X: the underlying set is the set of closed subsets of X, equipped with the representation ψ^+ defined by:

$$\psi^+(p) = A \iff [[\rho \to c_{\mathbb{S}}] \to c_{\mathbb{S}}](p) = \{O \in \mathcal{O}(X),\ O \cap A \neq \emptyset\}.$$

This representation is also often called the *representation of closed sets with positive information*.

The problem Overt Choice was introduced (as *Choice*) by Brattka and Hertling [6], and it has become one of the standard problems in computable analysis following [12]. Hoyrup introduced in [20] a variation of it called Π^0_2 Overt Choice. The following problem, Local Overt Choice[2], is similar to this last problem, but we consider only sets that are the intersection of a closed set given by overt information with an open set (given as a characteristic function).

Definition 2.7 Let $\mathbf{X}$ be a represented space. *Local Overt Choice* is the following problem:

$$\begin{aligned} \mathrm{OVC}_{\mathbf{X}} :\subseteq \mathcal{V}(\mathbf{X}) \times \mathcal{O}(\mathbf{X}) &\rightrightarrows X \\ (V, U) &\mapsto V \cap U, \end{aligned}$$

with $\mathrm{dom}(\mathrm{OVC}_{\mathbf{X}}) = \{(V, U) \mid V \cap U \neq \emptyset\}$.

[2] The relevance of this problem, which appears in Figure 1.1, was indicated to us by Arno Pauly.

2.6 Computably sequential subset

Let (X, ρ) be a represented space and $Y \subseteq X$ a subset of it. It is easy to see that the following map is computable:

$$\begin{aligned}\mathcal{O}(\mathbf{X}) &\to \mathcal{O}(\mathbf{Y}) \\ U &\mapsto U \cap Y\end{aligned}$$

By [54], this map is surjective exactly when the subset topology on Y is sequential.

Definition 2.8 We say that Y is a *computably sequential subset* of (X, ρ), or that $(Y, \rho_{|Y})$ is *computably sequentially embedded* in (X, ρ), if the projection $\mathcal{O}(\mathbf{X}) \to \mathcal{O}(\mathbf{Y})$ has a computable multivalued inverse.

This notion was introduced by Bauer [3] in the context of synthetic topology under the name *intrinsic subset*.

3 Notions of bases

We consider five notions of effective bases/subbases for a represented space $\mathbf{X} = (X, \rho)$. Let $\mathfrak{B}$ be a subset of $\mathcal{O}(X)$, and β a representation of $\mathfrak{B}$.

3.1 Semi-effective base

Definition 3.1 We say that $(\mathfrak{B}, \beta)$ is a *semi-effective base* if the map $\mathfrak{B} \hookrightarrow \mathcal{O}(X)$ is computable and if $\mathfrak{B}$ is a base of $\mathcal{O}(X)$.

Thus, the elements of $\mathfrak{B}$ are uniformly open, but the assumption that $\mathfrak{B}$ is a base is purely a classical one.

3.2 Nogina base

Definition 3.2 We say that $(\mathfrak{B}, \beta)$ is a *Nogina base* if it is a semi-effective base and if furthermore the following multi-function is computable:

$$\begin{aligned}N :\subseteq X \times \mathcal{O}(X) &\rightrightarrows \mathfrak{B} \\ (x, O) &\mapsto \{B \in \mathfrak{B}, x \in B \subseteq O\}\end{aligned}$$

Here, $\mathrm{dom}(N) = \{(x, O), x \in O\} \subseteq X \times \mathcal{O}(X)$.

This notion of base is closely related to Nogina [46, 47], this explains our choice of name. It is similar to Spreen [58], however Spreen uses more restrictive conditions based on a *strong inclusion relation*. This notion is called "pointwise base" in Bauer [2]. It was also used in [16], in the computably enumerable version, under the name "effective countable base".

The following straightforward proposition relates the above definition to conditions on the base $\mathfrak{B}$ that can be found in [46, 47].

Proposition 3.3 *If $(\mathfrak{B}, \beta)$ is a Nogina base, then the following multifunction is computable:*

$$N' :\subseteq X \times \mathfrak{B} \times \mathfrak{B} \rightrightarrows \mathfrak{B}$$
$$(x, B_1, B_2) \mapsto \{B \in \mathfrak{B}, x \in B \subseteq B_1 \cap B_2\}$$

Here, $\operatorname{dom}(N') = \{(x, B_1, B_2), x \in B_1 \cap B_2\}$.

3.3 Lacombe base

Consider a semi-effective base $(\mathfrak{B}, \beta)$ for $\mathbf{X} = (X, \rho)$. The following fact is well known (see for instance [49, Corollary 10.2], or [4]).

Lemma 3.4 *The computable injection $i : \mathfrak{B} \hookrightarrow \mathcal{O}(X)$ yields a computable function*

$$j : \begin{cases} \mathcal{V}(\mathfrak{B}) & \to \mathcal{O}(X) \\ A & \mapsto \bigcup_{b \in A} i(b) \end{cases}$$

Proof We have the following equivalence, for $x \in X$ and $A \in \mathcal{V}(\mathfrak{B})$:

$$x \in \bigcup_{b \in A} i(b) \iff \exists b \in A,\ x \in i(b).$$

Note that the condition $x \in i(b)$ defines a computable map $X \times \mathfrak{B} \to \mathbb{S}$, because $i : \mathfrak{B} \hookrightarrow \mathcal{O}(X)$ is computable. But, by effective currying, a map of type $X \times \mathfrak{B} \to \mathbb{S}$ is computable if and only if the corresponding map $X \to \mathcal{O}(\mathfrak{B})$ is computable. We thus have a computable map $\hat{\imath} : X \to \mathcal{O}(\mathfrak{B})$.

By definition of $\mathcal{V}(\mathfrak{B})$, $\exists : \mathcal{V}(\mathfrak{B}) \times \mathcal{O}(\mathfrak{B}) \to \mathbb{S}$ given by $(A, O) \mapsto (\exists x \in A \cap O)$ is computable.

Because $(x, A) \mapsto \exists b \in A,\ x \in i(b)$ is the composition of $\exists : \mathcal{V}(\mathfrak{B}) \times \mathcal{O}(\mathfrak{B}) \to \mathbb{S}$ with $\hat{\imath} : X \to \mathcal{O}(\mathfrak{B})$, it is indeed a computable map on $\mathcal{V}(\mathfrak{B}) \times X \to \mathbb{S}$, i.e., on $\mathcal{V}(\mathfrak{B}) \to \mathcal{O}(X)$. □

Definition 3.5 We say that the semi-effective base $(\mathfrak{B}, \beta)$ is a *Lacombe base* if the map $j : \mathcal{V}(\mathfrak{B}) \to \mathcal{O}(X)$ is onto, and if it has a computable multivalued right inverse: a computable $\psi : \mathcal{O}(X) \rightrightarrows \mathcal{V}(\mathfrak{B})$ such that $j \circ \psi = \mathrm{id}_{\mathcal{O}(X)}$.

In other words, open sets of X can uniformly be written as overt unions of basic sets.

The name Lacombe base comes from Lachlan [40] and Moschovakis [44] in reference to [41], where Lacombe introduced the idea that the computably open sets would be computable unions of basic open sets. This notion is called "pointfree base" in [2]. It was also considered in [4] or in [12].

Classically, the set of all open sets of a topology is always a base for this topology. The effective version of this fact is the following statement.

Proposition 3.6 *For any represented space, the identity on $\mathcal{O}(X)$ is a Lacombe base.*

Proof We know by Lemma 3.4 that there is a computable map $\mathcal{V}(\mathcal{O}(X)) \to \mathcal{O}(X)$. We have to show that it has a computable multivalued inverse. There is a computable map $\mathcal{O}(X) \to \mathcal{V}(\mathcal{O}(X))$ given by $U \mapsto \overline{\{U\}}$ [49, Proposition 7.4(2)]. We show that

$$U = \bigcup_{b \in \overline{\{U\}}} b.$$

This follows directly from the fact that the closure of $\{U\}$ in the Scott topology is exactly $\{V \in \mathcal{O}(X),\ V \subseteq U\}$. Indeed, if $V \subseteq U$, then $V \in \overline{\{U\}}$, since Scott open sets are upper sets. And if $V \not\subseteq U$, then there is some $x \in V \setminus U$, and V belongs to the Scott open $\mathcal{N}_x = \{O \in \mathcal{O}(X),\ x \in O\}$, while U does not, so $V \notin \overline{\{U\}}$. □

3.4 Representation subbase

Let X be a set, $\mathfrak{B} \subseteq \mathcal{P}(X)$ a set of subsets of X (which we think of as a subbase for a topology), and $\beta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathfrak{B}$ a representation of $\mathfrak{B}$.

Suppose that:

(1) For every x in X, the set $\mathcal{N}_x^{\mathfrak{B}} = \{B \in \mathfrak{B} \mid x \in B\}$ is open in the final topology of β.

(2) For every x and y in X, $x \neq y$ implies $\mathcal{N}_x^{\mathfrak{B}} \neq \mathcal{N}_y^{\mathfrak{B}}$. (Or again: $\mathfrak{B}$ is a subbase for a T_0 topology.)

Then we can define a representation of X, which we denote by β^*, by the following formula:

$$\forall p \in \mathbb{N}^{\mathbb{N}},\ \beta^*(p) = x \iff [\beta \to c_{\mathbb{S}}](p) = \mathcal{N}_x^{\mathfrak{B}}.$$

In other words: the β^*-name of a point x is a name of the open set $\mathcal{N}_x^{\mathfrak{B}}$. Or again: a point is described its properties.

The representation β^* is called the *subbase representation associated to* $(\mathfrak{B}, \beta)$.

Building on work of Schröder and relying on a Galois connection implicitly present in [54], we prove in [8] the following:

Theorem 3.7 *A representation of X is admissible (resp. computably admissible) if and only if it is continuously equivalent (resp. computably equivalent) to a representation of the form β^*.*

Definition 3.8 We say that $(\mathfrak{B}, \beta)$ is a *representation subbase of* (X, ρ) if $\mathfrak{B}$ is (classically) a subbase of the final topology of ρ, and if furthermore $\rho \equiv \beta^*$.

In the computably second countable case, when $\mathfrak{B}$ can be taken to be a totally numbered base, the assumption that it is a subbase of the final topology of ρ follows automatically from the equivalence $\rho \equiv \beta^*$ (Corollary 6.2). But in general, the final topology of β^* can be strictly finer than the topology generated by the subbase $\mathcal{B}$. This is precisely the topic of [8], and we refer the interested reader to this paper for more details.

As an immediate consequence of Theorem 3.7, we get:

Proposition 3.9 *A represented space has a representation subbase if and only if it is computably Kolmogorov.*

3.5 Enumeration subbase

The following notion is a generalization of the "standard representation" considered by Weihrauch, Kreitz and Grubba in [39, 62, 63, 66].

The idea that a proper generalization of the standard representation uses a type of formal inclusion relation goes back to Spreen [59], see also [51].

The following construction applies only to second-countable spaces, but the represented base we consider does not have to be countable.

Consider a represented base $(\mathfrak{B}, \beta)$. Consider a relation $\prec$ on $\mathrm{dom}(\beta)$ which is a *c.e. strong inclusion relation* [58, 59], i.e., a relation on $\mathrm{dom}(\beta)$ which satisfies the following two conditions:

(1) The relation $\prec$ is transitive and semi-decidable: $\prec: \mathrm{dom}(\beta) \times \mathrm{dom}(\beta) \to \mathbb{S}$ is computable.

(2) For all p, q in $\mathrm{dom}(\beta)$, $p \prec q \implies \beta(p) \subseteq \beta(q)$.

We will make two further assumptions on the relation $\prec$. A *formal neighborhood base* for a point x is a subset N_x of $\mathrm{dom}(\beta)$ that satisfies the two conditions:

- $\forall b \in N_x,\ x \in \beta(b)$,
- $\forall b_1 \in \mathrm{dom}(\beta),\ x \in \beta(b_1) \implies \exists b_2 \in N_x,\ b_2 \prec b_1$.

We will assume:

(3) Every point of X admits a formal neighborhood base.

Note that if $\prec$ is reflexive, points will automatically all admit formal neighborhood bases. Finally, we add the following condition, which ensures the T_0 condition:

(4) Points are uniquely determined by their formal neighborhood bases.

In other words, if N is a formal neighborhood base of x and y, then $x = y$.

We then define a representation $\beta^{\prec}$ of X by the following:

$$\beta^{\prec}(\langle u_0, u_1, \dots \rangle) = x \iff \{u_i, i \in \mathbb{N}\} \text{ is a formal neighborhood base of } x.$$

(Here $\langle \cdot \rangle : \Pi_{i \in \mathbb{N}} \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a countable tupling function.)

This representation can be thought of as a generalization of the Cauchy representation of metric spaces: the name of a point is a list of names of basic open sets that close in on this point.

Definition 3.10 We say that $(\mathfrak{B}, \beta)$ is an *enumeration subbase for* (X, ρ) if $\rho \equiv \beta^{\prec}$.

Example 3.11 Consider $\mathbb{R}$ with the Cauchy representation, equipped with the base that consists of all open intervals. Consider the representation of this base that comes from the Cauchy representation of $\mathbb{R}$, and a strong inclusion relation given by $]x, y[\prec]z, t[\iff x > z \,\&\, y < t$. Then this base is an enumeration subbase.

Note that we cannot ask, as it is the case in the standard Kreitz–Weihrauch representation, to have the name of a point x to be a list of all names of basic open sets that contain it, since a name has to be a countable sequence.

Proposition 3.12 *If $(\mathfrak{B}, \beta)$ is an enumeration subbase for (X, ρ), then the natural inclusion $\mathfrak{B} \hookrightarrow \mathcal{O}(X)$ is computable.*

Proof We want to show that from the $\beta^{\prec}$-name of a point x and the β-name b_1 of a basic set B it is possible to semi-decide whether $x \in B$. But $x \in B$ if and only if there appears in the name of x some β-name b_2 with $b_2 \prec b_1$. This is semi-decidable because $\prec$ is. □

Proposition 3.13 *If $(\mathfrak{B}, \beta)$ is an enumeration subbase for (X, ρ), then $\mathfrak{B}$ is (classically) a subbase for $\mathcal{O}(X)$.*

Proof By Proposition 3.12, we know that the elements of $\mathfrak{B}$ are open. Let $x \in X$ and $O \in \mathcal{O}(X)$, with $x \in O$. Then a finite prefix of the name of x already determines that $x \in O$, since the membership relation is open in $X \times \mathcal{O}(X)$.

This prefix intersects, via the tupling function $\langle \cdot \rangle$, finitely many β-names: $u_1, \dots, u_k$. Thus, x belongs to the finite intersection $\beta(u_1) \cap \dots \cap \beta(u_n)$, and we must have $\beta(u_1) \cap \dots \cap \beta(u_n) \subseteq O$, and $\mathfrak{B}$ is indeed a subbase. □

A useful property of representations coming from enumeration bases is that they are open:

Proposition 3.14 *The representation $\beta^{\prec}$ is open.*

Proof Let $w \in \mathbb{N}^*$, we consider the set $\beta^{\prec}(w\mathbb{N}^{\mathbb{N}})$.

The representation $\beta^{\prec}$ is defined thanks to a tupling function $\langle \cdot \rangle$. Notice that there exists a tuple $(w_1, \dots, w_k)$ of elements of $\mathbb{N}^*$ such that

$$w\mathbb{N}^{\mathbb{N}} = \langle w_1\mathbb{N}^{\mathbb{N}}, w_2\mathbb{N}^{\mathbb{N}}, \dots, w_k\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}} \dots \rangle.$$

Then we have

$$\beta^{\prec}(w\mathbb{N}^{\mathbb{N}}) = \bigcap_{i \le k} \bigcup_{p \in w_i\mathbb{N}^{\mathbb{N}} \cap \mathrm{dom}(\beta)} \beta(p),$$

which is a finite intersection of unions of open sets, thus open. □

4 Relation between the different notions of bases

Here, we establish the implication relations that relate the different notions of (sub)bases introduced in Section 3. In Section 5, we provide counterexamples which show that no other relations hold.

Note that implications between notions of subbases and notions of bases are obtained by replacing the subbase by the base it induces, and using the naturally induced representation.

Definition 4.1 If $(\mathfrak{B}, \beta)$ is a represented subbase, define the *induced represented base* $(\cap\mathfrak{B}, \cap\beta)$ by:

- $\cap\mathfrak{B}$ is the set of finite intersections of elements of $\mathfrak{B}$,
- The representation $\cap\beta$ is given by:

$$\mathrm{dom}(\cap\beta) = \{\langle k, u_0, \dots, u_k \rangle \in \mathbb{N}^{\mathbb{N}}, k \in \mathbb{N}, \forall i \le k,\ u_i \in \mathrm{dom}(\beta)\};$$
$$\forall \langle k, u_0, \dots, u_k \rangle \in \mathrm{dom}(\cap\beta),\ \cap\beta(\langle u_0, \dots, u_k \rangle) = \bigcap_{0 \le n \le k} \beta(u_n).$$

If $\prec$ was a c.e. strong inclusion for β, we define the induced strong inclusion on $\mathrm{dom}(\cap\beta)$ as follows:

$$\langle u_0, \dots, u_k \rangle \prec \langle v_0, \dots, v_{k'} \rangle \iff \forall i \le k', \exists j \le k,\ u_j \prec v_i.$$

It is easy to check that this relation remains c.e., and that all points still admit formal neighborhood bases if this was already the case.

The following shows that the operation of replacing a subbase by a base is harmless from the point of view of the notions of effective (sub)bases given in Section 3. Its proof is straightforward and left to the reader.

Proposition 4.2 *Let* $(\mathfrak{B}, \beta)$ *be a represented subbase for* $\mathbf{X}$*, and* $(\cap\mathfrak{B}, \cap\beta)$ *the base it induces. Then* $\beta^* \equiv (\cap\beta)^*$ *and* $\beta^{\prec} \equiv (\cap\beta)^{\prec}$*.*

We now proceed to prove the part of Theorem A that concerns implications between notions of represented bases.

The implications Nogina base $\implies$ Semi-effective base $\implies$ Classical base are straightforward.

The implication Representation subbase $\implies$ Semi-effective base is also trivial (modulo taking the induced base), since the definition of the representation β^* immediately guarantees that the map $\mathfrak{B} \to \mathcal{O}(X)$ will be computable.

Proposition 4.3 *An enumeration subbase yields a Nogina base.*

Proof Given a pair (x, U), with $x \in U$, run the program that defines U with, as input, the name of x. This must terminate. Once this computation has ended, only a finite portion of the name of x was visited: this finite portion contains the beginning of the names of finitely many basic open sets whose intersection will serve as a witness to the Nogina condition. □

Proposition 4.4 *A Lacombe base of a computably Kolmogorov space is a representation subbase.*

Proof Let $\mathfrak{B}$ be the base and denote by $i : \mathfrak{B} \hookrightarrow \mathcal{O}(X)$ the associated computable injection.

We know that the name of $\mathcal{N}_x = \{O \in \mathcal{O}(X),\ x \in O\} \in \mathcal{O}(\mathcal{O}(X))$ can be converted into a ρ-name of x.

We have to show that a name of $\mathcal{N}_x^{\mathfrak{B}} = \{B \in \mathfrak{B},\ x \in i(B)\} \in \mathcal{O}(\mathfrak{B})$ can also be converted into a ρ-name of x, this gives a priori less information, since $i(\mathfrak{B})$ does not contain all open sets.

But by hypothesis the Sierpiński representation of $\mathcal{O}(X)$ is equivalent to the representation associated to overt unions of basic sets.

For an overt $A \subseteq \mathfrak{B}$, $x \in \bigcup_{b \in A} i(b) \iff \exists b \in A,\ x \in i(b) \iff \exists b \in A,\ b \in i^{-1}(\mathcal{N}_x) \iff \exists b \in A \cap \mathcal{N}_x^{\mathfrak{B}}$. This last condition defines a computable map $\mathcal{V}(\mathfrak{B}) \times \mathcal{O}(\mathfrak{B}) \to \mathbb{S}$. Thus, by the smn theorem a name of $\mathcal{N}_x^{\mathfrak{B}} \in \mathcal{O}(\mathfrak{B})$ can be translated into a name of the map

$$\begin{aligned} \mathcal{V}(\mathfrak{B}) &\to \mathbb{S} \\ A &\mapsto x \in \bigcup_{b \in A} i(b). \end{aligned}$$

This is what was to be shown. □

Proposition 4.5 *A Lacombe base with computable Local Overt Choice is a Nogina base.*

Proof Let $\mathfrak{B}$ be the base and denote by $i : \mathfrak{B} \hookrightarrow \mathcal{O}(X)$ the associated computable injection. Let $\mathrm{OVC}_{\mathfrak{B}}$ be the local overt choice function associated to $\mathfrak{B}$: given an overt set and an open set that intersect, it produces a point in their intersection.

For $x \in X$, we define $\mathcal{N}_x^{\mathfrak{B}} = \{B \in \mathfrak{B} \mid x \in B\}$. The map $x \mapsto \mathcal{N}_x^{\mathfrak{B}}$ is easily seen to be computable. The Nogina condition is then simply expressed as the composition of the multifunction $\mathcal{O}(X) \rightrightarrows \mathcal{V}(\mathfrak{B})$ which allows us to express open sets of X as overt unions of basic set, composed with the following multi-function:

$$\begin{aligned} X \times \mathcal{V}(\mathfrak{B}) &\rightrightarrows \mathfrak{B} \\ (x, A) &\mapsto \mathrm{OVC}_{\mathfrak{B}}(A, \mathcal{N}_x^{\mathfrak{B}}). \end{aligned}$$

Both multi-functions are, by hypothesis, computable. □

5 Counterexamples that separate notions of represented bases

In Section 7, we provide counterexamples that separate notions of c.e. bases. Thus, it is sufficient here to separate notions of bases that are non-equivalent in general, but that become equivalent in the case of c.e. bases.

5.1 A representation subbase that is not a Nogina base

We now give an example of a representation subbase that does not satisfy the Nogina condition. In particular, it cannot be an enumeration subbase.

Consider $X = K \oplus K^c = \{2k, k \in K\} \cup \{2k+1, k \notin K\}$. Take $\mathfrak{B} = \{\{n\}, n \in X\}$ with the numbering $\beta :\subseteq \mathbb{N} \to \mathfrak{B}$ defined by $\beta(n) = \{n\}$, for $n \in X$.

We consider the subbase representation β^* associated to β: the β^*-name of a point n is a Sierpiński name of $\{\{n\}\}$.

Recall that an explicit model of the Sierpiński representation associated to a representation ρ is as follows [49]: a sequence $(u_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ is the name of an open set A if and only if u_0 is the code of a Type 2 machine that stops exactly on ρ-names of elements of A, and $(u_n)_{n\geq 1}$ is the oracle that this machine requires to operate.

For each $i \in \mathbb{N}$, consider the code t_i for a Type 2 machine that accepts exactly $2i$ and $2i+1$, and requires no oracle to function. Then $t_i 0^\omega$ (the concatenation of t_i with the constant zero sequence) is a valid β^*-name, representing either $2i$ or $2i+1$, depending on whether $i \in K$ or $i \in K^c$.

Suppose that the Nogina condition holds for β^* and β, and let N be a computable realizer of the Nogina condition: if n_x is the name of a point x in X and n_A is the name of some open set A with

$x \in A$, then $N(n_x, n_A)$ is the β-name of a basic set $\{n\}$ with $x \in \{n\} \subseteq A$. And so $n = x$, and also $N(n_x, n_A) = x$.

Let n_X be a computable Sierpiński name of X.

The map

$$\begin{aligned} \tilde{N} : \mathbb{N} &\to \mathbb{N} \\ i &\mapsto N(t_i 0^\omega, n_X) \end{aligned}$$

is then computable, and for all $i \in \mathbb{N}$, we have $\tilde{N}(i) = 2i$ if $i \in K$, and $\tilde{N}(i) = 2i + 1$ if $i \in K^c$. This is a contradiction, and the Nogina condition cannot hold for β^* and β.

5.2 A representation subbase that is not a Lacombe base

Recall (Theorem A) that a Lacombe base $\mathfrak{B}$ which admits a computable local overt choice is automatically a Nogina base. We show that the example given above (see Section 5.1) does admit a computable local overt choice. Because we have shown that this base is not a Nogina base, this will imply that it is not a Lacombe base either.

The base was $\mathfrak{B} = \{\{n\}, n \in X\}$ equipped with the numbering $\beta :\subseteq \mathbb{N} \to \mathfrak{B}$ defined by $\beta(n) = \{n\}$, for $n \in X$.

What we will show is that the representation of overt subsets of $\mathfrak{B}$ is equivalent to the representation where a set is given by an enumeration (with a pause symbol) of its elements. It is clear that this representation allows for a computable local overt choice, and thus this is sufficient to conclude.

The name of an overt subset A of $\mathfrak{B}$ encodes a program which, given an open set U of $\mathfrak{B}$, stops if and only U intersects A.

Notice that, for each $n \in \mathbb{N}$, the program that accepts exactly n defines an open subset U_n of $\mathfrak{B}$: it is empty if $n \notin X$, and $\{n\}$ otherwise.

Thus, given an overt set A, it is possible to list those U_n which intersect A, this will precisely give an enumeration of the elements of A.

5.3 An enumeration subbase that does not induce a Lacombe base

Here we show that an enumeration subbase does not automatically yield a Lacombe base.

In the same setting as above, we will consider a different representation.

Consider a non-c.e. subset A of $\mathbb{N}$. Consider the numbering $\beta :\subseteq \mathbb{N} \to \mathfrak{B}$ defined by $\beta(n) = \{n\}$, for $n \in A$.

The representation $\beta^{\subseteq}$ that follows from the definition of an enumeration subbase by taking actual inclusion as a formal inclusion is equivalent to the natural numbering ρ of A induced by the identity on $\mathbb{N}$:

$$\mathrm{dom}(\rho) = A;$$

$$\forall n \in \mathrm{dom}(\rho),\ \rho(n) = n.$$

The ρ-c.e. open sets are just sets of the form $A \cap E$, where E is an arbitrary c.e. subset of $\mathbb{N}$ (indeed, A is automatically a computably sequential subset of $\mathbb{N}$). Yet the computable unions of basic sets are *c.e. subsets* of A, these form in general a strict subset of the sets of the form $A \cap E$, for E c.e. For instance take $A \subseteq \mathbb{N}$ to be $K^c \oplus K$. Then $A \cap 2\mathbb{N}$ is not a c.e. subset of A (because it is not c.e.), yet it is the intersection of a c.e. set with A.

5.4 An enumeration subbase which is not a representation subbase

Putting together the two examples above works: we have a single numbered base which gives different representations by taking either the enumeration definition or the representation subbase definition.

6 Notions of effective second countability

6.1 Main results on computably second countable spaces

For each of the notions of effective base described in Section 3, we get a notion of effective second countability, by replacing the represented base $\beta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathfrak{B}$ by a base equipped with a total numbering: $\beta : \mathbb{N} \to \mathfrak{B}$. We will see that notions of bases that were not equivalent become equivalent.

Note first that the general notion of Lacombe base relies on overt unions. But on $\mathbb{N}$, the representations of overt subsets and of open sets coincide with the "enumeration representation", where a set is given by an enumeration that has access to a pause symbol (so that the empty set can be given by an enumeration). This simplifies the handling of this representation.

We now state our theorem on computably second countable spaces.

Theorem 6.1 *The following are equivalent:*

(1) (X, ρ) *has a totally numbered enumeration subbase.*

(2) (X, ρ) *is a represented space with a representation* ρ *which is computably admissible and equivalent to a computably open representation.*

(3) (X, ρ) *has a totally numbered representation subbase.*

(4) (X, ρ) *has a totally numbered Lacombe base and is computably Kolmogorov.*

Note that the implication (1) $\implies$ (4) can be seen as a Type 2 equivalent of a theorem of Moschovakis [44] set in Markovian computable analysis, and which proves that the Nogina and Lacombe approaches agree on recursive Polish spaces. The hypotheses required to prove the Type 1 and Type 2 theorems are different: computable separability is central in [44] (see the proof given in [50]) whereas it plays no role here.

Proof (1) $\implies$ (2) Let $\beta^{\prec}$ be the representation associated to an enumeration subbase $(B_n)_{n\in\mathbb{N}}$. The image of $u_0u_1\dots u_k\mathbb{N}^{\mathbb{N}}$ by $\beta^{\prec}$ is the set $B_{u_0}\cap\dots\cap B_{u_k}$. A name of this set as an element of $\mathcal{O}(X)$ can be computed from $u_0u_1\dots u_k$ by Proposition 3.12. Thus, $\beta^{\prec}$ is computably open.

We now show that $\beta^{\prec}$ is computably admissible. Notice that the subbase representation associated to $(B_n)_{n\in\mathbb{N}}$ translates to $\beta^{\prec}$: for any $x\in X$, given a program that halts exactly on those i such that $x\in B_i$, it is possible to enumerate the set $\{i\in\mathbb{N},\, x\in B_i\}$, this enumeration yields a name of x for $\beta^{\prec}$.

Conversely, by Proposition 3.12, the map $\in\colon X\times\mathbb{N}\to\mathbb{S}$, $(x,n)\mapsto x\in B_n$ is computable. This implies that, given a $\beta^{\prec}$-name for a point x, it is possible to compute a name of this point with respect to the subbase representation associated to $(B_n)_{n\in\mathbb{N}}$.

Thus, the representation $\beta^{\prec}$ is equivalent to the subbase representation associated to $(B_n)_{n\in\mathbb{N}}$. And this representation yields a CT_0 space by Theorem 3.7.

(2) $\implies$ (3) Suppose that ρ is computably open. The set $B=\{\rho(w\mathbb{N}^{\mathbb{N}}), w\in\mathbb{N}^*\}$, equipped with its natural numbering obtained via a numbering of $\mathbb{N}^*$, is a totally numbered semi-effective base of X. We show that it is even a representation subbase.

We must show that a Sierpiński name of the set

$$\mathcal{N}_x^B=\{w\in\mathbb{N}^*,\, x\in\rho(w\mathbb{N}^{\mathbb{N}})\}\in\mathcal{O}(\mathbb{N}^*)$$

can be translated into a ρ-name of x.

By assumption ρ is computably admissible, and thus any Sierpiński name of $\mathcal{N}_x^{\mathcal{O}(X)}=\{O\in\mathcal{O}(X),\, x\in O\}$ can be translated into a ρ-name of x.

It thus suffices to show that a name of $\mathcal{N}_x^B$ can be translated to a name of $\mathcal{N}_x^{\mathcal{O}(X)}$.

By currying, this is equivalent to showing that the map

$$\begin{aligned}\Theta :\subseteq \mathcal{O}(\mathbb{N}^*)\times\mathcal{O}(X)&\to\mathbb{S}\\ (\mathcal{N}_x^B, O)&\mapsto x\in O\end{aligned}$$

is computable (here $\mathrm{dom}(\Theta) = \{U \in \mathcal{O}(\mathbb{N}^*), \exists x \in X, U = \mathcal{N}_x^B\} \times \mathcal{O}(X)$). Fix $x \in X$ and $O \in \mathcal{O}(X)$.

Given a name of the set $\mathcal{N}_x^B$, it is possible to computably enumerate all finite prefixes of ρ-names of x, denote by $(w_i)_{i\in\mathbb{N}}$ the obtained sequence.

The Sierpiński name of O can be seen as a Type 2 machine that accepts ρ-names of points of O. If x belongs to O, for any ρ-name of x, a certain finite prefix of this name is already accepted by the machine. Therefore,

$$x \in O \iff \exists i,\ w_i \text{ is accepted by the machine encoded in the name of } O.$$

This is indeed a semi-decidable condition, and Θ is indeed computable.

(3) $\implies$ (1) Take $(B_n)_{n\in\mathbb{N}}$ a representation subbase. In this case, the ρ-name of a point x is given by the element $\{n \in \mathbb{N}, x \in B_n\} \in \mathcal{O}(\mathbb{N})$.

By taking equality of names as a strong inclusion for $(B_n)_{n\in\mathbb{N}}$, i.e., $b_1 \prec b_2 \iff b_1 = b_2$, we see that the result follows immediately from the fact that the Sierpiński representation on $\mathcal{O}(\mathbb{N})$ is equivalent to the following representation:

$$\begin{aligned} \tau : \mathbb{N}^{\mathbb{N}} &\to \mathcal{O}(\mathbb{N}) \\ p &\mapsto \{p_n - 1, n \in \mathbb{N} \,\&\, p_n > 0\}. \end{aligned}$$

(1) + (3) $\implies$ (4) A representation subbase always yields a computable Kolmogorov space.

Replace the subbase by the base it induces, by adding all finite intersections, and making the intersection map computable. We obtain a numbered base $(B_n)_{n\in\mathbb{N}}$.

We show how, given a name of $O \in \mathcal{O}(X)$ with respect to the Sierpiński representation, we can effectively construct a list L of names of basic sets such that $\bigcup_{n\in L} B_n = O$.

We can suppose that the representation ρ of X is an enumeration representation with respect to $(B_n)_{n\in\mathbb{N}}$.

Apply the program P given by the name of O to all finite sequences w, $w \in \mathbb{N}^*$. Whenever $w = v_0 \dots v_k$ is accepted, add a name of the intersection $B_{v_0} \cap \dots \cap B_{v_k}$ to L.

Remark that an element w is not necessarily the beginning of a valid name. If it is, and it is accepted by P, it must be that indeed the corresponding intersection $B_{v_0} \cap \dots \cap B_{v_k}$ is a subset of O. If on the contrary w is not the beginning of a valid name, it means exactly that $B_{v_0} \cap \dots \cap B_{v_k} = \emptyset$, and thus it will not affect the union that is being constructed. This shows that $\bigcup_{n\in L} B_n \subseteq O$.

Suppose now that $x \in O$, and take a name p of it. It must be accepted by P, and thus also some finite prefix of it, say w. Thus, w defines an intersection $B_{v_0} \cap \dots \cap B_{v_k}$ which was added to the list L, and thus $x \in \bigcup_{n\in L} B_n$. This gives the reverse inclusion.

(4) $\implies$ (3) This implication was already true in the general case of represented bases. □

By definition, a representation subbase for a represented space (X, ρ) is a classical subbase $\mathcal{B}$ equipped with a representation β such that $\rho \equiv \beta^*$. A corollary of the above proof is that, when considering a totally numbered set $\mathcal{B}$, the equivalence $\rho \equiv \beta^*$ automatically implies that $\mathcal{B}$ is a subbase of the topology of X (which is the final topology of ρ).

Corollary 6.2 *Let $(B_i)_{i\in\mathbb{N}}$ be a totally numbered set of subsets of X, which satisfies the conditions (1)-(2) of Section 3.4, so that B induces a representation B^* of X. Then $(B_i)_{i\in\mathbb{N}}$ is a subbase for the final topology of B^*.*

Proof One can immediately see that the proof of (3) $\implies$ (1) of Theorem 6.1 does not use the assumption that $(B_i)_{i\in\mathbb{N}}$ is (classically) a subbase for the topology of X. Thus, $(B_i)_{i\in\mathbb{N}}$ is automatically an enumeration subbase, and Proposition 3.13 shows that an enumeration subbase is automatically a classical subbase. □

As another corollary, we get:

Theorem 6.3 *Let (X, ρ) be a computably second countable represented space, let $Y \subseteq X$ be a subset of X, and equip it with the induced representation $\rho_{|Y}$. Then $(Y, \rho_{|Y})$ is also computably second countable, and Y is a computably sequential subset of (X, ρ).*

Proof Let $(B_n)_{n\in\mathbb{N}}$ be a representation subbase for (X, ρ). Consider the base $(B_n \cap Y)_{n\in\mathbb{N}}$. We claim that this is a representation subbase of $(Y, \rho_{|Y})$.

For any $y \in Y$, a realizer of the map

$$\begin{aligned}\mathcal{N}_y^X : \mathbb{N} &\to \mathbb{S}\\ n &\mapsto (y \in B_n)\end{aligned}$$

is the same thing as a realizer of

$$\begin{aligned}\mathcal{N}_y^Y : \mathbb{N} &\to \mathbb{S}\\ n &\mapsto (y \in B_n \cap Y).\end{aligned}$$

Thus, a name of $\mathcal{N}_y^Y$ can be seen as a name of $\mathcal{N}_y^X$, which can be turned into a ρ-name of y in X (as (X, ρ) is computably second countable), and then seen as a $\rho_{|Y}$-name of y.

Consequently, $(Y, \rho_{|Y})$ is computably second countable.

By Theorem 6.1, any O in $\mathcal{O}(Y)$ can be written as a countable union

$$\bigcup_{n\in A}(B_n \cap Y),$$

with $A \in \mathcal{O}(\mathbb{N})$. This union can also be seen as the union

$$\left(\bigcup_{n \in A} B_n\right) \cap Y.$$

But the union $\bigcup_{n \in A} B_n$ is an element of $\mathcal{O}(X)$, and thus we have uniformly expressed any element of $\mathcal{O}(Y)$ as the intersection of Y with an element of $\mathcal{O}(X)$. □

The following proposition generalizes the "computably open representation" characterization of computable second countability.

Proposition 6.4 *Let $f : X \to Y$ be a map between* CT_0 *represented spaces X and Y. If f is computable, computably open and surjective, and X is computably second-countable, then so is Y.*

Proof Let $(B_n)_{n \in \mathbb{N}}$ be a Lacombe base for X. We show that $(f(B_n))_{n \in \mathbb{N}}$ is a Lacombe base for Y. Because f is computably open, the map $\mathbb{N} \to \mathcal{O}(Y)$, $n \mapsto f(B_n)$ is indeed computable. Given an open set $U \subseteq Y$, $f^{-1}(U)$ can be written as a union

$$f^{-1}(U) = \bigcup_{i \in I} B_i$$

for a set $I \in \mathcal{O}(\mathbb{N})$ which can be computed from a name of U. Because f is onto, we have $U = f(f^{-1}(U))$, and so

$$U = f\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f(B_i).$$

Thus, $(f(B_n))_{n \in \mathbb{N}}$ is indeed a Lacombe base. □

The fact that a represented space is computably second countable if and only if it computably embeds into $\mathcal{O}(\mathbb{N})$ is well known, it can be found for instance in [33, Section 2.4.1]. This completes the set of implications about the different notions of effective second countability that appear in Theorem A.

Note finally that a representation ρ is computably fiber-overt if and only if it is computably open [34] (a representation ρ of X is *computably fiber-overt* if the map $X \to \mathcal{V}(\mathbb{N}^{\mathbb{N}})$, $x \mapsto \rho^{-1}(x)$ is well-defined and computable). Computably fiber-overt representations appear for instance in [7, 12, 48].

6.2 Additional properties of computably second countable spaces

In the context of computably second countable spaces, several important properties can be characterized in terms of properties related to a base $(B_n)_{n \in \mathbb{N}}$.

Proposition 6.5 *The following are equivalent for a represented space* $\mathbf{X}$*:*

(1) $\mathbf{X}$ *is computably second countable and overt,*

(2) $\mathbf{X}$ *is* CT_0*, and it admits a Lacombe base* $(B_n)_{n\in\mathbb{N}}$ *which does not contain the empty set.*

Proof We first prove (1) $\implies$ (2). By Theorem 6.1, we can suppose that $\mathbf{X}$ is CT_0 and has a Lacombe base $(B_n)_{n\in\mathbb{N}}$. But this base could contain the empty set. However, by overtness, there is a procedure which selects those basic sets which are not empty. Denote by $\hat{B_i}$ the i-th element of $(B_n)_{n\in\mathbb{N}}$ which is found to be non-empty by this procedure. We claim that $(\hat{B_n})_{n\in\mathbb{N}}$ is also a Lacombe base of $\mathbf{X}$. Indeed, any element of $\mathcal{O}(\mathbf{X})$ can be written as a union of the basic sets $(B_n)_{n\in\mathbb{N}}$, and by construction it is immediate to see that such a union can computably be converted in a union of the basic sets $(\hat{B_n})_{n\in\mathbb{N}}$.

We now prove the converse implication. Suppose that $(B_n)_{n\in\mathbb{N}}$ is a Lacombe base that does not contain the empty set. In order to prove that an open set is non-empty, it suffices to write is as the union of a set of basic sets given by an enumeration. Such an enumeration uses a special symbol to indicate that nothing is enumerated at certain stages, and an open set given in this way is non-empty if and only if a basic set appears in this enumeration, this is indeed semi-decidable. □

Computable points, sets and functions have simple characterizations in the context of computably second countable spaces:

Proposition 6.6 *Let* $\mathbf{X}$ *be a computably second-countable space with base* $(B_i)_{i\in\mathbb{N}}$*. Then:*

(1) $x\in\mathbf{X}$ *is computable* $\iff \{n\in\mathbb{N}\mid x\in B_n\}$ *is c.e.*

(2) $U\subseteq\mathbf{X}$ *is c.e. open* $\iff U=\bigcup_{i\in I}B_i$ *for some c.e. set* I*.*

(3) $A\subseteq\mathbf{X}$ *is overt* $\iff \{n\in\mathbb{N}\mid A\cap B_n\neq\emptyset\}$ *is c.e.*

(4) $K\subseteq\mathbf{X}$ *is computably compact* $\iff \{(n_1,\dots,n_k)\in\mathbb{N}^*\mid K\subseteq\bigcup_{i=1}^k B_{n_i}\}$ *is c.e.*

(5) *For any represented space* $\mathbf{Y}$ *and function* $f:Y\to X$*, the following are equivalent:*
   - $f^{-1}:\mathcal{O}(X)\to\mathcal{O}(Y),\ U\mapsto f^{-1}(U)$ *is computable,*
   - $(f^{-1}(B_n))_{n\in\mathbb{N}}$ *is a computable sequence of c.e. open sets of* Y*.*

(6) *If* $\mathbf{Y}$ *is also computably second countable with base* $(D_i)_{i\in\mathbb{N}}$*, the above is also equivalent to:*
   - *There is a computable function* $p:\mathbb{N}^2\to\mathbb{N}$ *that satisfies:* $\forall n\in\mathbb{N}, f^{-1}(B_n)=\bigcup_{i\in\mathbb{N}}D_{p(i,n)}$.

Proof All statements are immediate consequences of Theorem 6.1. □

The interest of Proposition 6.6 lies in the fact that all the right-hand side statements have historically been used as *definitions* (for computable points, computably open sets, effectively continuous functions, and so on). See for instance Lacombe [41], Ceitin [13], Spreen [58], and Iljazović and Sušić [25]. Here, however, we interpret them not as definitions, but as simple characterizations that hold in the context of computably second-countable spaces.

If $\mathbf{X}$ is computably second countable, then the represented space $\mathcal{V}(\mathbf{X})$ of closed overt subsets and the represented space $\mathcal{K}(\mathbf{X})$ of compact subsets of $\mathbf{X}$ are also computably second countable.

Theorem 6.7 *Let $\mathbf{X}$ be a computably second countable represented space with base $(B_i)_{i\in\mathbb{N}}$. Then, $\mathcal{V}(\mathbf{X})$ and $\mathcal{K}(\mathbf{X})$ are also computably second countable. Indeed, the following maps define totally numbered representation subbases for $\mathcal{V}(\mathbf{X})$ and $\mathcal{K}(\mathbf{X})$ respectively:*

(1) $\diamond B:\mathbb{N}\to\mathcal{O}(\mathcal{V}(\mathbf{X})),\ n\mapsto\{A\in\mathcal{V}(\mathbf{X})\mid A\cap B_n\neq\emptyset\}$.

(2) $\square B:\mathbb{N}^*\to\mathcal{O}(\mathcal{K}(\mathbf{X})),\ (n_1,\dots,n_k)\mapsto\{K\in\mathcal{K}(\mathbf{X})\mid K\subseteq\bigcup_{i=1}^k B_{n_i}\}$.

The topology on $\mathcal{V}(\mathbf{X})$ is the lower Fell topology and the topology of $\mathcal{K}(\mathbf{X})$ is the upper Vietoris topology.

Proof The result follows directly from the following equivalences, together with Theorem 6.1:

- For $A\subseteq X$ closed and $U\subseteq X$ open, with $U=\bigcup_{i\in I}B_i$ for some $I\subseteq\mathbb{N}$, we have
$$A\cap U\neq\emptyset\iff\exists n\in\mathbb{N},\ A\cap B_i\neq\emptyset.$$
- For $K\subseteq X$ compact and $U\subseteq X$ open, with $U=\bigcup_{i\in I}B_i$ for some $I\subseteq\mathbb{N}$, we have
$$K\subseteq U\iff\exists(n_1,\dots,n_k)\in\mathbb{N}^*,\ A\subseteq\bigcup_{i=1}^k B_{n_i}.$$ □

Of course, $\mathbf{X}$ can be computably second countable without $\mathcal{O}(\mathbf{X})$ being (computably) second countable: this fails already for $\mathbf{X}=\mathbb{N}^{\mathbb{N}}$, as it is well known that $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ is not second countable. Something else can be said of $\mathcal{O}(\mathbf{X})$ when $\mathbf{X}$ is supposed to be computably second countable: its representation is computably equivalent to a total representation. Having a total representation is a form of completeness for represented spaces (in particular, it coincides with completeness for metric spaces [5] and quasi-metric spaces [9]).

Proposition 6.8 *Let $\mathbf{X}$ be a computably second countable represented space. Then $\mathcal{O}(\mathbf{X})$ admits a total representation.*

Proof The representation of open sets associated to a Lacombe base $(B_i)_{i\in\mathbb{N}}$ is total. □

Selivanov studied in [56] total representations of $\mathcal{O}(X)$ for a second countable space X (see Theorem 8.6 in [56]). But his results do not involve computability.

7 Counterexamples that separate notions of effective second countability

Proposition 7.1 *The c.e. open sets of an admissibly represented space do not have to generate its topology.*

Proof We consider a total numbering ν of $\{0,1\}$, given by $\nu(n) = 1 \iff n \in \mathrm{Tot}$, where Tot designates $\{n \in \mathbb{N}, \mathrm{dom}(\varphi_n) = \mathbb{N}\}$.

The numbering ν, seen as a representation, is admissible for the discrete topology on $\{0,1\}$. However, it is immediate to see that the only c.e. open sets are the empty set and the whole set itself. □

An example similar to the above one can be extracted from [23, Theorem 10].

We now turn to another separation result needed to establish Theorem A.

Proposition 7.2 *The c.e. open sets of a* CT_0 *represented space can generate its topology without it being semi-effectively second countable.*

We consider an effective version of a well known example of a sequential but not Fréchet–Urysohn space. This example was studied by Schröder in [54] as an easy example of an admissibly represented space which is not second countable.

We first describe the classical example. Consider the set X given by:

$$X = \mathbb{N}^2 \cup (\{\infty\} \times \mathbb{N}) \cup \{(\infty, \infty)\}.$$

For m_0, n_0 in $\mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, denote by

$$D_{n_0,m_0} = \{(n, m_0), n \geq n_0, n \in \mathbb{N} \cup \{\infty\}\};$$

$$E_{m_0,f} = \{(\infty, \infty)\} \cup \bigcup_{m \geq m_0} D_{f(m),m}$$

Define also:

$$\mathfrak{B}_1 = \{\{(n, m)\}, n, m \in \mathbb{N}\},$$

$$\mathfrak{B}_2 = \{D_{n_0,m_0}, n_0, m_0 \in \mathbb{N}\},$$

$$\mathfrak{B}_3 = \{E_{m_0,f}, m_0 \in \mathbb{N}, f \in \mathbb{N}^{\mathbb{N}}\}.$$

Put $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3$, and consider the topology on X generated by $\mathfrak{B}$.

Note that the set $\mathfrak{B}_3$ is not countable. And in X, while (∞,∞) is adherent to $\mathbb{N}^2$, no sequence of points of $\mathbb{N}^2$ converges to (∞,∞), because this would require that its second component should grow faster that *any* function $f:\mathbb{N}\to\mathbb{N}$.

Here we consider an effective version of this construction: we replace $\mathfrak{B}_3$ by $\mathfrak{B}_3^+$:

$$\mathfrak{B}_3^+ = \{E_{m_0,f}, m_0 \in \mathbb{N}, f \in \mathbb{N}^{\mathbb{N}} \text{ is a total computable function}\}$$

Denote by $\mathfrak{B}^+ = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3^+$. We define a numbering β of $\mathfrak{B}^+$ by

$$\beta(\langle 1,n,m\rangle) = \{(n,m)\},$$

$$\beta(\langle 2,n,m\rangle) = D_{n,m},$$

$$\beta(\langle 3,m,i\rangle) = E_{m,\varphi_i} \text{ if } \varphi_i \text{ is a total function.}$$

To the numbered base $(\mathfrak{B}^+,\beta)$ we associate the representation ρ of X which is just the subbase representation. This guarantees that (X,ρ) is CT_0.

We will now prove that this space is the desired counterexample to prove Proposition 7.2.

Say that a point x is *effectively adherent* to a set A if the following multi-function is computable:

$$\Theta_{x,A} :\subseteq \mathcal{O}(X) \rightrightarrows X$$
$$O \mapsto A \cap O,$$

with $\mathrm{dom}(\Theta_{x,A}) = \{O \in \mathcal{O}(X),\ x \in O\}$. Notice the following easy result:

Proposition 7.3 *If a represented space (X,ρ) is semi-effectively second countable, then any point that is effectively adherent to a set is the limit of a computable sequence of elements of this set.*

Lemma 7.4 *The point (∞,∞), while effectively adherent to $\mathbb{N}^2$, is not the limit of a computable sequence of points of $\mathbb{N}^2$.*

Proof We first show that (∞,∞) is effectively adherent to $\mathbb{N}^2$. But this is obvious: any neighborhood of (∞,∞) contains points of $\mathbb{N}^2$, one of these can be found by exhaustive search.

Now we show that (∞,∞) is not the limit of a computable sequence of points of $\mathbb{N}^2$.

This follows immediately from the fact that if a sequence $((u_n,v_n))_{n\in\mathbb{N}} \subseteq \mathbb{N}^2$ converges to (∞,∞), then we should have that for every computable function f and every set $E_{m_0,f}$, $((u_n,v_n))_{n\in\mathbb{N}}$ eventually belongs to $E_{m_0,f}$. This implies that eventually $u_n \geq f(v_n)$ for every computable function f. This is not possible for a computable sequence. □

Proposition 7.2 follows easily.

Proof of Proposition 7.2 This is simply Proposition 7.3 together with Lemma 7.4. □

Proposition 7.5 *A represented space can have a totally numbered Lacombe base without being* CT_0.

Proof Consider $\{0, 1\}$ with the discrete topology and the representation $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ given by

$$\rho((u_n)_{n\in\mathbb{N}}) = u_0 \bmod 2,$$

and $\mathrm{dom}(\rho) = \{u \in \mathbb{N}^{\mathbb{N}}, u \text{ is not computable}\}$. This representation is admissible: a translation from the usual representation ν of $\{0, 1\}$ can be computed by any non-computable oracle.

We first show that the base given by $B_0 = \{0\}$ and $B_1 = \{1\}$ is a Lacombe base for $(\{0, 1\}, \rho)$.

This follows directly from the fact that the Sierpiński representation associated to ρ is equivalent to the Sierpiński representation associated to ν: $[\rho \to c_{\mathbb{S}}] \equiv [\nu \to c_{\mathbb{S}}]$.

To prove this, it suffices to show that the map

$$\begin{aligned} \in : \{0, 1\} \times \mathcal{O}(\{0, 1\}) &\to \mathbb{S} \\ (x, O) &\mapsto x \in O \end{aligned}$$

is $(\nu \times [\rho \to c_{\mathbb{S}}], c_{\mathbb{S}})$-computable (instead of only $(\rho \times [\rho \to c_{\mathbb{S}}], c_{\mathbb{S}})$-computable, as would be expected).

Consider the $[\rho \to c_{\mathbb{S}}]$-name of an open set O. It consists of a pair (u, v), where $u \in \mathbb{N}$ is the code of a Type 2 machine which, when run with access to the oracle $v \in \mathbb{N}^{\mathbb{N}}$, will accept exactly the ρ-names of points of O. Now it is easy to see that $0 \in O$ if and only if some finite sequence $0w \in \mathbb{N}^*$ is accepted by this machine, and $1 \in O$ if and only if some finite sequence $1w \in \mathbb{N}^*$ is accepted by this machine. These conditions being semi-decidable, this indeed shows that $\in : \{0, 1\} \times \mathcal{O}(\{0, 1\})$ is $(\nu \times [\rho \to c_{\mathbb{S}}], c_{\mathbb{S}})$-computable.

Finally, ρ is not computably admissible: if it was, by Theorem 6.1, the base (B_0, B_1) would be a representation subbase, but the representation induced by the base (B_0, B_1) is precisely the natural representation ν of $\{0, 1\}$, which is not computably equivalent to ρ. □

The following proposition contains a represented space which is very close to being computably second countable -but which still is not.

Proposition 7.6 *A computably admissibly represented space can be Nogina second countable without being Lacombe second countable.*

Proof We define a representation β of a base for the discrete topology on $\mathbb{N}$ by the following:

$$\beta(k^\omega) = \{k\},\ k \in \mathbb{N},$$

$$\beta(\mathbf{1}_K) = \mathrm{Tot},$$

where $\mathbf{1}_K$ is the characteristic function of the halting set. Thus, the above is almost the obvious base for the discrete topology on $\mathbb{N}$, but we artificially add a $\emptyset'$-computable name to Tot.

Let ρ denote the subbase representation induced by β. It is thus computably admissible.

Notice that $\rho \leq \mathrm{id}_{\mathbb{N}}$. The ρ-name of a point n can be thought of as a pair, consisting of n together with a method to determine whether or not n belongs to Tot when given access to an oracle for the halting set K.

Let $\tilde{\mathbb{N}}$ be the represented space $(\mathbb{N}, \rho)$.

One immediately checks that the base $\{\{k\}, k \in \mathbb{N}\}$ is a Nogina base for $\tilde{\mathbb{N}}$. Thus, $(\mathbb{N}, \rho)$ is Nogina second countable.

We now show that $\tilde{\mathbb{N}}$ does not have a totally numbered Lacombe base.

We first consider another base $(B_i)_{i\in\mathbb{N}}$ for the discrete topology on $\mathbb{N}$, given by $B_0 = \mathrm{Tot}$, $B_i = \{i-1\}$ for $i > 0$. Consider the associated subbase representation $\hat{\rho}$. The represented space $(\mathbb{N}, \hat{\rho})$ is, by construction, computably second countable. By Theorem 6.1, the Sierpiński representation $[\hat{\rho} \to c_{\mathbb{S}}]$ is equivalent to the "union representation" associated to $(B_i)_{i\in\mathbb{N}}$: open sets can uniformly be written as countable unions of basic sets. Denote by $\cup B$ this last representation.

The map $i \mapsto B_i$ is a numbering whose image is identical to the image of β, it can be seen as a representation which we denote by B. We then have $\beta \leq B$, and because the map $\rho \mapsto [\rho \to c_{\mathbb{S}}]$ is order reversing for translation of representations, it follows that $\hat{\rho} \leq \rho$ and $[\rho \to c_{\mathbb{S}}] \leq [\hat{\rho} \to c_{\mathbb{S}}]$.

Suppose that there exists a Lacombe base $(\mathfrak{B}_i)_{i\in\mathbb{N}}$ for $\tilde{\mathbb{N}}$, and denote by $\cup\mathfrak{B}$ the representation of open sets associated to countable unions of elements of $\mathfrak{B}$. We get:

$$\mathfrak{B} \leq \cup\mathfrak{B} \equiv [\rho \to c_{\mathbb{S}}] \leq [\hat{\rho} \to c_{\mathbb{S}}] \equiv \cup B.$$

By construction of ρ, Tot has an $\emptyset'$-computable $[\rho \to c_{\mathbb{S}}]$-name, and thus it also admits an $\emptyset'$-computable $\cup\mathfrak{B}$-name, which we call p_{Tot}. Translating this name to $\cup B$, we get a $\emptyset'$-computable name of Tot for $\cup B$. But it is clear that a $\cup B$-name of Tot either contains explicitly the B-name 0 for Tot, or it is not $\emptyset'$-computable. Thus, the $\emptyset'$-computable $\cup B$-name of Tot contains 0. This implies that a finite prefix w_{Tot} of p_{Tot} is mapped, via the realizer of the translation $\cup\mathfrak{B} \leq \cup B$, to a finite prefix of a $\cup B$-name that contains 0. In turn, the prefix w_{Tot} can be extended to a computable $\cup\mathfrak{B}$-name of Tot, for instance the sequence $(w_{\mathrm{Tot}})^\omega$ must be a valid $\cup\mathfrak{B}$-name for Tot.

Thus, we get that Tot is actually a c.e. open set for ρ.

We conclude by proving that this is impossible.

For each $i \in \mathbb{N}$, consider a code t_i for the program that, on input k, runs $\varphi_i(i)$ for k computations steps, and halts if this computation does not stop during those k steps. Otherwise, it loops indefinitely.

There is a computable function Φ that transforms t_i into a ρ-name of t_i. Indeed, because, by construction, $t_i \in \mathrm{Tot} \iff i \notin K$, there is an algorithm which, given the β-name $\mathbf{1}_K$ of Tot, decides whether or not $t_i \in \mathrm{Tot}$.

Thus, if Tot were c.e. open for ρ, there would exist a program which, given i, would stop if and only if $t_i \in \mathrm{Tot}$. Via the above equivalence this would give a program to semi-decide the complement of K. □

8 Schröder's Metrization theorem as a sharp theorem

Classically, the following equivalences hold for a metric space (X, d):

(1) (X, d) is separable,

(2) (X, d) (topologically) embeds into the Hilbert cube $[0, 1]^{\mathbb{N}}$,

(3) (X, d) is second countable.

Furthermore, the Urysohn Metrization Theorem [67] shows that a second countable topological space is metrizable if and only if it is regular and T_1, or equivalently normal and T_1.

The notion of "effective regularity" that was found out by Schröder [52] to be the correct one to establish a metrization theorem is not the first one one naturally would think of. Different notions of effective regularity were introduced and compared by Weihrauch in [65]. Note however that in [65] all spaces are supposed to be computably second countable.

Definition 8.1 A represented space (X, ρ) is *computably regular* if the following multi-function is well-defined and computable:

$$R :\subseteq X \times \mathcal{A}_-(X) \rightrightarrows \mathcal{O}(X)^2$$
$$(x, A) \mapsto \{(U, V),\ x \in U \,\&\, A \subseteq V \,\&\, U \cap V = \emptyset\},$$

where $\mathrm{dom}(R) = \{(x, A),\ x \notin A\}$.

A represented space (X, ρ) is *strongly computably regular* if the following multi-function is well-defined and computable:

$$P : \mathcal{O}(X) \rightrightarrows \mathcal{O}(X)^{\mathbb{N}} \times \mathcal{A}_-(X)^{\mathbb{N}}$$
$$O \mapsto \{(U_n, V_n)_{n\in\mathbb{N}},\ \forall n \in \mathbb{N},\ U_n \subseteq V_n \subseteq O,\ O = \bigcup_{n\in\mathbb{N}} U_n\}.$$

A represented space is *computably normal* if the following multi-function is well-defined and computable:

$$S :\subseteq \mathcal{A}_-(X) \times \mathcal{A}_-(X) \rightrightarrows \mathcal{O}(X) \times \mathcal{O}(X)$$
$$(A,B) \mapsto \{(U,V),\ A \subseteq U, B \subseteq V, U \cap V = \emptyset\}.$$

Here, $\mathrm{dom}(S) = \{(A,B),\ A \cap B = \emptyset\}$.

Note the following lemma, which gives sufficient conditions to go from computable regularity to strong computable regularity:

Lemma 8.2 ([65]) *On overt and computability second countable represented spaces, computable regularity is equivalent to strong computable regularity.*

The effective Urysohn lemma states:

Lemma 8.3 (Effective Urysohn Lemma, [52]) *On a computably normal space, the following multi-function is computable:*

$$R :\subseteq \mathcal{A}_-(X) \times \mathcal{A}_-(X) \rightrightarrows \mathcal{C}(X,\mathbb{R})$$
$$(A,B) \mapsto \{f, A \subseteq f^{-1}(0),\ B \subseteq f^{-1}(1)\}.$$

Here again, $\mathrm{dom}(R) = \{(A,B), A \cap B = \emptyset\}$.

Recall that a represented space (X,ρ) *computably embeds* into a represented space (Y,τ) if there is a (ρ,τ)-computable injection $X \hookrightarrow Y$ which admits a (τ,ρ)-computable partial inverse.

We now prove the following (which is a slight modification of a result used in [1]):

Theorem 8.4 (Schröder–Urysohn Effective Metrization) *The following are equivalent for a represented space (X,ρ):*

(1) (X,ρ) *computably embeds into the Hilbert cube,*

(2) (X,ρ) *computably embeds into some computable metric space,*

(3) (X,ρ) *is computably second countable and strongly computably regular.*

The above imply, but are not equivalent to:

(4) (X,ρ) *is computably second countable and admits a computable metric that generates its topology.*

Proof (1) $\implies$ (2) is clear.

(2) $\implies$ (3) Being computably second countable is inherited by subsets, and so is strong computable regularity.

(3) $\implies$ (1) The first step in the proof given in [52] is to prove that strong computable regularity and computable second countability imply computable normality, and thus that the Effective Urysohn Lemma applies.

Then, the proof given by Schröder [52] consists in building a computable double sequence of functions $g_{i,j} : X \to [0,1]$ that separates points, i.e., such that for all $x, y \in X$ with $x \neq y$, $g_{i,j}(x) \neq g_{i,j}(y)$ for some $(i,j) \in \mathbb{N}^2$.

The functions $g_{i,j}$ are defined as follows.

Fix $(B_i)_{i\in\mathbb{N}}$, the countable base that witnesses computable second countability. By strong computable regularity, there is a computable double sequence $(U_{i,j}, A_{i,j})_{(i,j)\in\mathbb{N}^2} \in \mathcal{O}(X)^{\mathbb{N}} \times \mathcal{A}_{-}(X)^{\mathbb{N}}$ such that:

$$\forall i \in \mathbb{N},\ B_i = \bigcup_{j\in\mathbb{N}} U_{i,j},$$

$$\forall (i,j) \in \mathbb{N}^2,\ U_{i,j} \subseteq A_{i,j} \subseteq B_i.$$

Then, apply the Effective Urysohn's lemma to B_i^c and $A_{i,j}$: this gives a computable function $g_{i,j}$ such that $B_i^c \subseteq g_{i,j}^{-1}(\{1\})$ and $A_{i,j} \subseteq g_{i,j}^{-1}(\{0\})$.

In [52] (and in the classical Urysohn theorem), a metric d is defined as

$$d(x,y) = \sum_{i,j} 2^{-\langle i,j\rangle} |g_{i,j}(x) - g_{i,j}(y)|.$$

Here, as in [1], we consider the map $h(x) = (g_{i,j}(x))_{\langle i,j\rangle\in\mathbb{N}}$. It is by construction a computable map from (X,ρ) to the Hilbert cube. What we have to show is that it is injective and admits a computable inverse.

Let y be a point in $[0,1]^{\mathbb{N}} \cap \mathrm{Im}(h)$. Let $x = h^{-1}(y)$ be a preimage of y. Let B_i be some basic set of X.

Notice that by construction, for every $j \in \mathbb{N}$ and z in B_i^c, $g_{i,j}(z) = 1$. Thus, for every $j \in \mathbb{N}$ and every $z \in X$, $g_{i,j}(z) < 1 \implies z \in B_i$. But, as also follows from the construction of the functions $g_{i,j}$, it is also true that for every $z \in B_i$, there is some $j \in \mathbb{N}$ such that $g_{i,j}(z) < 1$: this holds whenever z belongs to the open set $U_{i,j}$.

Thus, we get the equivalence:

$$x \in B_i \iff \exists j \in \mathbb{N},\ g_{i,j}(x) < 1.$$

This equivalence immediately implies what was to be shown:

- Because X is T_0, the above equivalence shows that the map h is injective.
- And because the condition $\exists j \in \mathbb{N},\, g_{i,j}(x) < 1$ is semi-decidable in terms of a name of $y = h(x)$, this equivalence also shows that the map h has a computable inverse. Indeed, it shows that given the name of an element $y = h(x)$ of the Hilbert cube, it is possible to compute a name of $\mathcal{N}_x^B = \{i \in \mathbb{N},\, x \in B_i\} \in \mathcal{O}(\mathbb{N})$, and by computable second countability this name can be translated into a ρ-name of x.

Finally, (1) $\implies$ (4) is clear, and the fact that (4) $\not\implies$ (3) can be found in [65, Example 5.4] (see also [45], where Weihrauch's example is analyzed in details). □

9 Open choice, non-total open choice, overtness and separability

In this section, we study effective versions of the following classical fact:

Fact 9.1 *A second countable space is separable.*

Proof Consider a countable base (B_i). The set $\{B_i \mid B_i \neq \emptyset\}$ is also countable. Then apply choice. □

The effective version of separability is computable separability: a represented space is *computably separable* if it admits a dense and computable sequence.

It is easy to see that in order to obtain an effective version of the argument above, we will require overtness, to prove that the set of non-empty subsets of a computably enumerable base is also computably enumerable. We will also use a form of effective choice axiom, which we apply only to open sets. We could naively use the following choice problem:

Open choice:

$$\begin{aligned} OC : \mathcal{O}(X) \setminus \{\emptyset\} &\rightrightarrows X \\ O &\mapsto O. \end{aligned}$$

The naive effective version of Fact 9.1 is the following:

Proposition 9.2 *Let (X, ρ) be a semi-effectively second countable represented space which is overt and has a computable open choice problem. Then (X, ρ) is effectively separable.*

The above proposition is obviously true, but it is in fact completely uninteresting, because of the stronger result, which makes no second countability assumption:

Theorem 9.3 *A represented space (X,ρ) has computable open choice if and only if it is computably separable. In particular, a space with a computable open choice is overt.*

Proof The proof relies on the fact that $\{X\}$ is dense in $\mathcal{O}(X)$ for the Scott topology. We first show that this also happens at the level of names of open sets: if O is an open set of X, then there is a certain name of O which is in the closure of the set of names of X.

Suppose that $X \neq \emptyset$ (otherwise we have nothing to do). The subset of $\mathcal{O}(X)$ consisting of $\emptyset$ and of X is homeomorphic to the Sierpiński space $\mathbb{S}$. It is easy to see that this is effective: there is a computable embedding $\mathbb{S} \overset{e}{\hookrightarrow} \mathcal{O}(X)$ mapping $\top$ to X and $\bot$ to $\emptyset$. Denote by E a computable realizer of e.

It is also well known that the union map $(V,W) \overset{u}{\mapsto} V \cup W$ is computable on $\mathcal{O}(X)$. Let U be the natural computable realizer of u: given as input two open sets V and W, each represented by the code of a Type 2 machine together with the oracle that machine requires, U produces the code of a machine that halts if and only if one of these machines halts (and which uses as oracle the pairing of the two oracles).

For each name $s \in \{0^n1^\omega, n \in \mathbb{N}\} \cup \{0^\omega\}$ of an element of the Sierpiński space and each name p of an element O of $\mathcal{O}(X)$, we consider the name $U(E(s),p) \in \mathbb{N}^{\mathbb{N}}$.

If $s = 0^n1^\omega$, $U(E(s),p)$ is a name of $X \cup O = X$. If $s = 0^\omega$, $U(E(s),p)$ is a name of $\emptyset \cup O = O$.

The Sierpiński name of an element of $\mathcal{O}(X)$ is an encoded pair $\langle n,p\rangle$, $n \in \mathbb{N}$ and $p \in \mathbb{N}^{\mathbb{N}}$, where n is the code for a Type 2 Turing machine and p is the oracle that this machine will use.

Let $(w_n)_{n\in\mathbb{N}}$ be the computable sequence of all names of the form $\langle n,p\rangle$, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^{\mathbb{N}}$ an eventually constant sequence. These elements do not have to be valid names of elements of $\mathcal{O}(X)$, because we have not guaranteed extensionality: the Turing machine number n that uses p as oracle can accept some names of a point and reject others.

Consider now the computable double sequence $v_{n,t} = U(E(0^t1^\omega), w_n)$, for $n,t \in \mathbb{N}$. Each $v_{n,t}$ is a name of X as an element of $\mathcal{O}(X)$ -whether or not w_n was a valid name of an element in $\mathcal{O}(X)$.

Suppose that we have a computable open choice OC for X with computable realizer $\hat{\mathrm{OC}}$.

We claim that the computable double sequence $(\rho(\hat{\mathrm{OC}}(v_{n,t})))_{(n,t)\in\mathbb{N}^2}$ is dense in X.

First, notice that for each n,t, $\hat{\mathrm{OC}}(v_{n,t})$ indeed belongs to $\mathrm{dom}(\rho)$, since we are applying open choice to a certain name of X, which is non-empty. Now let O be any non-empty open set of X, let p be a name of O, and $q = U(E(0^\omega),p)$. Thus, q is another name of O, and the realizer $\hat{\mathrm{OC}}$ applied to q yields the ρ-name of a point x in O. By continuity of the representation ρ of X, $\hat{\mathrm{OC}}$ applied to names sufficiently close to q will also yield names of points in O. Such name must appear among $(v_{n,t})_{(n,t)\in\mathbb{N}^2}$. □

To introduce the correct effectivization of Fact 9.1, we rely on the following choice problem:

Non-total open choice:

$$OC^* : \mathcal{O}(X) \setminus \{\emptyset, X\} \rightrightarrows X$$
$$O \mapsto O.$$

We then have:

Proposition 9.4 *Let (X,ρ) be a semi-effectively second countable represented space which is overt and has a computable non-total open choice problem. Suppose furthermore that some totally numbered semi-effective base consists only of strict subsets of X.*

Then (X,ρ) is effectively separable.

Note that if X can be written as a strict union of two c.e. open sets and is semi-effectively second countable, then there is also an effective enumeration of a base which never enumerates X. Thus, the assumption that some enumerable base avoids X is mild.

Proposition 9.4 is immediate, but we have to check that it is interesting, by the following:

Proposition 9.5 *Having computable non-total open choice does not imply computable separability, even on* CT_0 *spaces.*

Proof Consider a set $A \subseteq \mathbb{N}$. Consider the following representation of A:

$$\rho(u) = n \iff \mathrm{Im}(u) \cap A = \{n\}.$$

Thus, the ρ-name of a point n of A is a list of natural numbers which intersects A exactly in n.

Note that ρ is computably admissible. Suppose that we have a name of

$$\mathcal{N}_n^{\mathcal{O}(A)} = \{O \in \mathcal{O}(A) : n \in O\}$$

as an element of $\mathcal{O}(\mathcal{O}(A))$, i.e., a name that encodes the characteristic function of $\mathcal{N}_n^{\mathcal{O}(A)}$: on input of the name of an element U of $\mathcal{O}(A)$, it halts if and only if $n \in U$. We have to recover a ρ-name of n. For each t in $\mathbb{N}$, consider the following program O_t: on input of a ρ-name, it accepts it if and only if it contains t. If $t \in A$, a code for this program is a name of the open set $\{t\}$ of $\mathcal{O}(A)$. It is not a valid name of an element of $\mathcal{O}(A)$ when $t \notin A$, because in this case, for any $n \in A$, O_t accepts some ρ-names of n and rejects others. When applying the name of $\mathcal{N}_n^{\mathcal{O}(A)}$ to the code of O_t, either $t \in A$, and then O_t is accepted if and only if $t = n$, or $t \notin A$, in which case O_t does not define an element of $\mathcal{O}(A)$ and the behavior of the realizer of $\mathcal{N}_n^{\mathcal{O}(A)}$ on input O_t is unspecified: either it never accepts it, or it accepts it. But in any case, if O_t is accepted the realizer of $\mathcal{N}_n^{\mathcal{O}(A)}$, then $t = n$

or $t \notin A$. Thus, an enumeration of the set $\{t, O_t$ is accepted by the realizer of $\mathcal{N}_n^{\mathcal{O}(A)}\}$ is a ρ-name of n: it contains n and possibly some numbers outside of A.

Furthermore, (A, ρ) has computable non-total open choice. Indeed, suppose that some Sierpiński name P of a set B with $\emptyset \subsetneq B \subsetneq A$ is given. We see P as being the characteristic function of B. We can apply P in parallel to the following names: $0^\omega, 1^\omega, 2^\omega, \dots$ Notice that if $k \in A$, then k^ω is a valid name of a point of A, and thus it should be accepted at some point if and only if it belongs to B. On the other hand, if $k \notin A$, then it is an invalid name, but any finite prefix of it could be completed either into the name of a point of B, or into the name of a point of $A \setminus B$ (which is non-empty by hypothesis). Thus, k cannot be accepted by P. Thus, the procedure we describe will end up correctly selecting a point of B.

In the following lemma, Lemma 9.6, we show that some choice of A guarantees that (A, ρ) is not computably separable. □

Lemma 9.6 *There exists $A \neq \emptyset$ so that (A, ρ) is not computably separable.*

Proof We guarantee that no computable function $g : \mathbb{N} \to A$ has dense image. Notice that a computable realizer of a computable function $g : \mathbb{N} \to A$ is a Type 2 machine that takes as input a single natural number and outputs a (necessarily computable) sequence of natural numbers. By the smn-Theorem, we can in fact see this realizer as a computable function $f : \mathbb{N} \to \mathbb{N}$ which satisfies that, for each n in $\mathbb{N}$, $W_{f(n)}$ is a ρ-name of $g(n)$. (Here $W = \mathrm{dom}(\varphi)$ is the usual numbering of c.e. subsets of $\mathbb{N}$.)

Our goal is thus to built A such that there does not exist a total computable function f such that:

(4)
$$\forall n \in \mathbb{N},\ |W_{f(n)} \cap A| = 1;$$
$$A \subseteq \bigcup_{n \in \mathbb{N}} W_{f(n)}.$$

We diagonalize against all computable functions in a non-effective way.

Suppose that $A \cap \{0, \dots, N_{i-1}\}$ was already constructed, guaranteeing that the above conditions do not hold for φ_k, $k = 0, \dots, i-1$.

Suppose that φ_i is total.

Suppose that for some n, $W_{\varphi_i(n)}$ contains two points $x_1 < x_2$ both greater than N_{i-1}. Then we add x_1 and x_2 to A, guaranteeing that (4) will not be satisfied for φ_i. We then choose x_2 to be N_i: $A \cap \{0, \dots, N_i\}$ will not change anymore.

Suppose now that for some n, $W_{\varphi_i(n)}$ contains a point already added to A below N_{i-1}, and a point x above N_{i-1}. Again we add x to A, and (4) is invalidated. Fix $N_i = x$.

Suppose that for some n, $W_{\varphi_i(n)}$ contains exactly one point x above N_{i-1}, and possibly some points below N_{i-1}, but that do not belong to A. We then add $x+1$ to A, and chose $N_i = x+1$. Thus, $W_{\varphi_i(n)} \cap A$ will be empty.

Finally, the remaining case is that for every n, $W_{\varphi_i(n)}$ contains only points below N_{i-1}. Then we add $N_{i-1}+1$ to A, and fix $N_i = N_{i-1}+1$.

It is easy to check that all cases are covered, and that the constructed A is not empty, since each case covered adds at least one point to A. □

10 Further notes on the literature

10.1 Two branches in computable topology

The study of computable topology in the context of Type 2 computability can be split in two main branches, which we call **B1** and **B2**:

B1 The study of computable topological spaces up to homeomorphisms,

B2 The study of computable topological spaces up to computable homeomorphisms.

Some surveys and important results in the branch **B1** are: [14, 19, 21, 36, 57]. Papers illustrating **B2** are: [10, 11, 16, 18, 49].

In terms of represented spaces, these two branches correspond to studying respectively:

B1 Representations up to continuous translation;

B2 Representations up to computable translation.

These two topics are fundamentally different in terms of their goals. Results that are about spaces up to homeomorphisms are results in topology, since topology is the domain of mathematics that studies spaces up to homeomorphism. Thus, we could summarize **B1** as:

B1 Computable topology understood as the application of results and methods coming from computability in order to study topology.

On the other hand, results about spaces up to computable homeomorphism are about spaces that have strictly more structure than a topology.

B2 Computable topology understood as a form of constructive topology, which provides a refinement of classical results in topology.

Depending on the precise topic studied in computable topology, there can be more or less overlap between these two branches.

The present paper's main topic is the study of different notions of effective second countability.

On this particular topic, there is in fact absolutely no overlap between the two branches **B1** and **B2** described above.

Indeed, the following was proved by Hoyrup, Melnikov and Ng:

Theorem 10.1 ([22]) *Every second countable space admits a representation that makes it computably second countable and overt.*

This theorem shows that:

- Up to homeomorphism, every second countable space satisfies the strongest version of effective second countability.
- Weaker notions of effective second countability are relevant only when spaces are considered up to computable homeomorphism.

Thus, the study of the different notions of effective second countability is a topic in which the two branches **B1** and **B2** are as disjoint as can be.

But this is not the case of all topics that are studied in computable topology.

A nice example, pointed out to us by the referee, is related to the Birkhoff–Kakutani Theorem, which states that, for second countable topological groups, Hausdorffness is a sufficient condition for metrizability.

The effective content of this theorem is studied in [36], where the following is proved:

Theorem 10.2 ([36, Theorem 1.1]) *For a Polish group G that is either abelian or locally compact, the following are equivalent:*

(1) *G has a computable topological presentation,*

(2) *G has a right-c.e. Polish presentation.*

Furthermore, in (2) the metric can be taken left-invariant. (Or right-invariant.)

Here, *having a computable topological presentation* is equivalent to what we call, in the present paper, being computably second countable and overt, and a *right-c.e. presentation of a Polish space X* is a dense sequence $(u_n)_{n\in\mathbb{N}}$ such that the function $\mathbb{N}^2 \to \mathbb{R}$, $(n,m) \mapsto d(u_n,u_m)$ is right computable.

The subtlety that needs to be pointed out here is the following:

- The theorem cited above is a result about spaces up to homeomorphism.
- Thus, by looking only at the statement of this theorem, one may feel that it has no content in terms of the branch **B2** of computable topology.
- However, in order to obtain this result about spaces up to homeomorphism, an effective version of the Birkhoff–Kakutani Theorem is obtained in [36], and this result is valid up to computable homeomorphism.

This illustrates the fact that the distinction between the branches **B1** and **B2** is sometimes thinner than one may expect.

The effective Birkhoff–Kakutani Theorem, as proved in [36], recast in the present's paper vocabulary, is the following:

Theorem 10.3 ([36, Theorem 3.2], Effective Birkhoff–Kakutani Theorem) *Let G be a represented group with computable group operations. Suppose that:*

- *G is computably second countable;*
- *Overt;*
- *And classically Hausdorff.*

Then G admits a right-computable left invariant metric which is compatible with the topological structure of G.

This theorem is very interesting and raises several questions, especially when compared to the Schröder–Urysohn Metrization Theorem presented in the present paper.

We quote three questions which seem to us particularly interesting.

It was shown by Weihrauch that, by assuming overtness, it is possible to considerably simplify the hypotheses of the Schröder–Urysohn Metrization Theorem, because computable regularity can replace strong computable regularity (see Proposition 8.2). We thus ask:

Problem 10.4 Is there a generalization of Theorem 10.3 that does not rely on overtness?

Theorem 10.3 relies on a non-effective separation property: the group is assumed Hausdorff, but not necessarily computably so. As a necessary consequence, the metric that is obtained is only right-computable.

On the other hand, in the Schröder–Urysohn Metrization Theorem, the regularity hypothesis is made effective, and the produced metric is computable.

Problem 10.5 Is there an effective Birkhoff–Kakutani Theorem that, by starting with a computably Hausdorff group, produces a computable metric?

Problem 10.6 Is there an effective Urysohn Metrization Theorem that, by starting with a regular but not computably regular space, can characterize those spaces that admit a right-computable metric?

10.2 On the work of Kalantari and Welch

An interesting question, raised by the referee, is to understand how the present paper relates to the work of Kalantari and Welch on computable topology, which constitutes a rather important body of work, see for instance [26, 27, 28, 29, 30, 31, 32].

This problem has already been addressed several times, in particular by Kalantari and Welch themselves, and we refer the interested reader to the last section of [27] for a summary of their view of the problem of integrating their approach into other approaches.

We want to add the following points:

- The approach of Kalantari and Welch to computable topology fits in the global framework of represented spaces, this is explained in [27].
- All the spaces that are considered by Kalantari and Welch are metrizable, being second countable and regular [27].
- A fundamental aspect of the approach of Kalantari and Welch to computable topology is the use of a certain representation of points associated to *sharp filters* of basic open sets, see Definition 10.7 below.
- On the other hand, it seems that Kalantari and Welch have never fixed a preferred representation of open sets on the spaces that they consider, and notions such as "c.e. open sets" are absent from [26, 27, 28, 29, 30, 31, 32].

The representation which is present throughout all the papers of Kalantari and Welch on computable topology is the following one.

Definition 10.7 Let X be a second countable space equipped with a numbered base $\mathfrak{B} = (B_i)_{i\in\mathbb{N}}$.

- A sequence $(B_{u_n})_{n\in\mathbb{N}}$ of basic sets defines *a sharp filter* if:
  (1) For all n we have $\overline{B_{u_{n+1}}} \subseteq B_{u_n}$;
  (2) For any two basic sets C and D of $\mathfrak{B}$, with $\overline{C} \subseteq D$, there is n so that either $B_{u_n} \cap C = \emptyset$ or $B_{u_n} \subseteq D$.
- A sharp filter $(B_{u_n})_{n\in\mathbb{N}}$ *converges* to a point x if

$$\{x\} = \bigcap_{n\in\mathbb{N}} B_{u_n}.$$

- The *Kalantari–Welch representation* associated to $\mathfrak{B}$ is the representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ given by
$$\delta((u_n)_{n\in\mathbb{N}}) = x \iff (B_{u_n})_{n\in\mathbb{N}} \text{ is a sharp filter converging to } x.$$

The representation of points considered by Kalantari and Welch is not always compatible with the representations considered in the present paper. Indeed, we have the following proposition:

Proposition 10.8 *There is a computable Polish space in which the Kalantari–Welch representation of points associated to the basis consisting of rational open balls gives strictly less information than the Cauchy representation of points. In fact, the distance function is not right-computable with respect to this representation.*

Proof We take a certain subset of $\mathbb{R}$. Denote by K the halting set: $K = \{n \in \mathbb{N},\ \varphi_n(n)\downarrow\}$. Consider the union
$$A = \bigcup_{n\in K}[n-1/4, n+1/4] \cup \bigcup_{n\notin K}\{n\}.$$
A is a computably separable subset of $\mathbb{R}$. Indeed, the set of all natural numbers, together with the set of rationals in $[n-\frac{1}{4}, n+\frac{1}{4}]$, for each n in K, constitutes a computable and dense subset of A. Being a computably separable closed subset of $\mathbb{R}$, it is a computable Polish space.

The Cauchy representation on A is simply the representation inherited from the embedding of A in $\mathbb{R}$: a point x is described by a sequence of rationals (r_n) with $|x - r_n| < 2^{-n}$ for each n.

Now we show that the Kalantari–Welch representation associated to open balls provides strictly less information than the Cauchy representation.

Suppose that there exists a Type 2 machine M that translates from the Kalantari–Welch representation to the Cauchy representation.

For $n \in \mathbb{N}$, we consider the behavior of M if its input is the constant sequence of open balls $B(n, 1/2), B(n, 1/2), B(n, 1/2), B(n, 1/2), \ldots$

- If $n \notin K$, then this is a valid name with respect to the Kalantari–Welch representation, since $B(n, 1/2) = \{n\}$. (Condition (2) of Definition 10.7 is trivially satisfied for a singleton.) And so M should start writing a Cauchy name of n, in particular it should, at some point, provide an approximation of n within $1/8$.
- If $n \in K$, since we do have $\overline{B(n, 1/2)} \subseteq B(n, 1/2)$, any finite sequence
$$B(n, 1/2), B(n, 1/2), \ldots, B(n, 1/2)$$
constitutes the beginning of a valid name for *any* point in $[n-\frac{1}{4}, n+\frac{1}{4}]$. And in this case the machine M can never provide an approximation of n within $1/8$, since this would necessarily exclude numbers that could later turn out to correspond to the named number.

Thus, we see that the machine M should produce an approximation of its input within $1/8$ exactly for those n that do not belong to K. This is a contradiction because the complement of K is not c.e. Thus, the Kalantari–Welch representation does not translate to the Cauchy representation.

One easily notices that the example above also implies that the distance function is not right-computable with respect to the Kalantari–Welch representation. □

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