Journal of Logic & Analysis 2:6a (2010) 1–3
ISSN 1759-9008

Addendum and Erratum to "Geometric spaces with no points"

ROBERT S. LUBARSKY

Abstract: Some comments and clarifications are made to the author's paper [1].

2000 Mathematics Subject Classification 03F50, 03C62, 03E70, 03E40 (primary)

Keywords: constructive analysis, point-free geometry, topological models, Riesz spaces

After the publication of [1], some penetrating and well justified questions were received from a colleague, making it clear that what is folklore to some can be unclear to others. This note is to clarify some of the background assumed in that first article. We will also take this opportunity to correct some misleading and one mistaken assertion in the same. (The mistake is in a comment; all the results are correct.)

Topological models are just like the better known Boolean-valued or forcing models, before modding out by "not not." Hence it might be useful to describe the situation in the context of classical forcing first.

A set in any forcing extension is given by a term in the forcing language of the ground model. A term is any set of the form $\{\langle p_i, \sigma_i\rangle \mid i \in I\}$, where p_i is a forcing condition (a member of the forcing partial order $\mathcal{P}$), σ_i a term (inductively), and I any index set. The generic object G is given by $\{\langle p, \hat{p}\rangle \mid p \in \mathcal{P}\}$, where $\hat{}$ is the embedding of the ground model into the terms: inductively, $\hat{x} = \{\langle \top, \hat{y}\rangle \mid y \in x\}$. G is characterized by the relation $p \Vdash$ "$\hat{p} \in G$".

Often G is identified with some other set easily definable from G. Perhaps it would be easiest to illustrated what's going on here via an example. Consider the simplest forcing partial order, Cohen forcing, with conditions $2^{<\omega}$, finite binary sequences. In a generic extension, G is a set of finite binary sequences. Typically, though, people work instead with $\bigcup G$, which is an infinite binary sequence. To a set theorist, $\bigcup G$ is easily definable from G, and in the other direction G is the set of proper initial segments of $\bigcup G$. It's usually more convenient to work with $\bigcup G$, and so set theorists take advantage of this simple inter-definability, and, by abuse of language, refer to $\bigcup G$ as the generic G.

Published: November 2010 doi: 10.4115/jla.2010.2.6a

To make the analogy with topological models tighter, the partial order $2^{<\omega}$ is a basis for the Cantor space 2^ω. The open set $O(p)$ corresponding to p is the set of sequences with p as an initial segment. p could be viewed as a name for $O(p)$. Using open sets instead of the partial order, G is then defined as $\{\langle O, \hat{O}\rangle \mid O$ is an open set of $2^\omega\}$, and is characterized by $O \Vdash$ "$\hat{O} \in G$", not by $O \Vdash$ "$G \in \hat{O}$", much less, as stated in the original paper (p. 5), by $O \Vdash$ "$G \in O$", which is not merely mistaken, but actually incoherent, as O is not a term in the language.

Even if the latter is the way it's usually best thought of.

That needs explanation. Still working classically with $2^{<\omega}$, note that G's alter ego $\bigcup G$ is a member of the topological space under consideration, 2^ω, and is characterized by $p \Vdash \bigcup G \in O(p)$, at least when properly interpreted. $\bigcup G$ is certainly not in $\hat{2^\omega}$, which is 2^ω as interpreted in the ground model M, also written as $(2^\omega)^M$, because $\bigcup G$ is not even in M. Rather, $\bigcup G \in (2^\omega)^{M[G]}$. Similarly, we could not say that $p \Vdash \bigcup G \in \hat{O(p)}$; rather, $p \Vdash \bigcup G \in O(p)$, with the latter $O(p)$ being the open set determined by p in the extension $M[G]$.

Now we're in a position to explain the mysterious, misleading, and sometimes even mistaken comments of [1]. In Theorem 6 (p. 4) of the original paper, where we're forcing (i.e. taking the topological model) over the complex numbers $\mathbb{C}$, we refer to the generic complex number as given by $O \Vdash$ "$G \in O$". Strictly speaking, the generic G is given by $O \Vdash$ "$\hat{O} \in G$". But the open sets of diameter less than ϵ cover the space, so the generic determines a new complex number (as a Dedekind cut). By simple inter-definability, this new number is itself called G. It is determined by $O \Vdash$ "$G \in O$", where the latter O is viewed as the interpretation in the extension of some ground-model description of O. For an example of what such a description might be, every open set of complexes is a union of countably many discs with rational center and radius, and so can be described by a sequence of such center-radius pairs $(\langle c_n, r_n\rangle)_{n\in\omega}$.

The story with the construction in section 3 of [1] is similar, but more complicated. It is always the case that there is a generic G determined by $O \Vdash$ "$\hat{O} \in G$". One expects a G', equidefinable with G, with O forcing G' to be in O as interpreted in the extension. While something along those lines is likely to be true, it is not always so straightforward as in the cases above. Section 3 is a case in point. By analogy with simpler instances, one might reasonably have guessed that the generic object G is roughly the same as a new member, there called H, of the topological space F, a finite, non-empty set of complex numbers. The import of Theorem 8 is that that does not work, that the definition of H as a set of points leads to the empty set.

Of course, there is still G, and to understand any particular topological model is to understand G. In some sense, G is (equidefinable with) a generalized member of the topological space; the challenge is to determine what "generalized" means. In this case, G is the same as the distance function of Proposition 11. (Since that distance function was derived from the Riesz space R, G can just as well be identified with R.) In the earlier paper, the distance function of the model was given; what remains to be done is to show that the generic can be recovered from the distance function. This is not hard. Given $A \in F$, consider the open neighborhood U of A in normal form with a positive piece of information O_x for each $x \in A$, with O_x a disc with center x and radius ϵ small compared to the distances between the points in A. On all of the points on the boundary of O_x, the distance function has a value less than 2ϵ. So some open set of U is within the circle of radius 2ϵ around that boundary point. The intersection of all those boundary points is exactly O_x. This recovers O_x from the distance function.

The situation is similar to the first construction of [2]. There the topological space is the set of infinite, bounded sequences of natural numbers. The generic is an infinite sequences of naturals, but it's not bounded. Rather, it's pseudo-bounded, a weakening of boundedness. Since classically boundedness and pseudo-boundedness are equivalent, the generic can be viewed as a new member of the space, as long as the space is understood as the set of pseudo-bounded sequences. Similarly, in the case of interest here, the generic cannot be viewed as a new member of the space of finite, non-empty sets of complex numbers. Instead, in the classical meta-theory, the finite, non-empty sets can be identified with the distance functions on $\mathbb{C}$, and the generic understood as a new distance function, interestingly with no corresponding set of points from which it measures the distance.

References

[1] Robert S. Lubarsky, Geometric spaces with no points, *Journal of Logic and Analysis*, Vol. 2:6 (2010); doi:10.4115/jla.2010.2.6.

[2] Robert S. Lubarsky, On the failure of BD-$\mathbb{N}$, submitted for publication

Florida Atlantic University, Dept. of Mathematical Sciences, 777 Glades Rd., Boca Raton, FL 33431, USA

Robert.Lubarsky@alum.mit.edu

Received: 3 October 2010 Revised: 31 October 2010