Journal of Logic & Analysis 3:4 (2011) 1–15
ISSN 1759-9008

Uniform liftings of continuous mappings

ALEXANDER P PYSHCHEV

Abstract: We investigate the question of when a continuous mapping between subspaces of nonstandard hulls has a uniform lifting.

2000 Mathematics Subject Classification 03H05, 54J05 (primary); 54C20, 54D20, 54D65 (secondary)

Keywords: nonstandard analysis, nonstandard hull, uniform lifting

1 Introduction

Let $\mathcal{H}$ and $\mathcal{H}'$ be (some) nonstandard hulls of *metric spaces (M, d) and (M', d') respectively and $\mathcal{M}$ a subspace of $\mathcal{H}$. Following Fajardo and Keisler [6, Definition 4.14], we say that a mapping $f : \mathcal{M} \to \mathcal{H}'$ is *uniformly liftable* if there exists an internal mapping $\phi : M \to M'$ such that for every equivalence class $\mu \in \mathcal{M}$ the image $\phi[\mu]$ is contained in $f(\mu)$; the mapping ϕ in this case is said to be a *uniform lifting* of f. It is easily checked that every uniformly liftable mapping is continuous. The converse is not true:

Example 1.1 Let d be a $\{0, 1\}$-valued *metric on ${}^*\mathbb{N}$ and let

$$\mathcal{H} = \{\{x\} : x \in {}^*\mathbb{N}\}$$

be the nonstandard hull of $({}^*\mathbb{N}, d)$. (Here and in the following $\mathbb{N}$ denotes the set of nonnegative integers.) It is clear that the mapping $f : \mathcal{H} \to \mathcal{H}$ defined by $f(\{x\}) = \{\chi_{\mathbb{N}}(x)\}$, where $\chi_{\mathbb{N}} : {}^*\mathbb{N} \to \{0, 1\}$ is the characteristic function of $\mathbb{N}$, is continuous but not uniformly liftable.

By [5, Proposition 4.12] and [6, Corollary 4.8, Theorem 4.18] every continuous mapping $f : \mathcal{M} \to \mathcal{H}'$ defined on a compact set $\mathcal{M}$ is uniformly liftable. More generally, we have the following result: (We assume that our nonstandard universe $(V(\Xi), V({}^*\Xi), *)$ is κ-saturated, where κ is an uncountable cardinal.)

Published: April 2011 doi: 10.4115/jla.2011.3.4

Proposition 1.2 *Let $f : \mathcal{M} \to \mathcal{H}'$ be a continuous mapping from a subspace $\mathcal{M}$ of the nonstandard hull $\mathcal{H}$ of a *metric space (M, d) around $c \in M$ into the nonstandard hull $\mathcal{H}'$ of a *metric space (M', d') around $c' \in M'$. If $\mathcal{M}$ has a dense subspace of cardinality $< \kappa$ and $|\mathcal{M}| < |V(\Xi)|$, then f is uniformly liftable.*

Proof Assume first that f is an isometric embedding of $\mathcal{M}$ into $\mathcal{H}'$. Let D be a dense subspace of $\mathcal{M}$ of cardinality $< \kappa$. For every $\mu \in D$ choose $x_\mu \in \mu$ and $y_\mu \in f(\mu)$. For every $(\mu, n) \in D \times \mathbb{N}$ let $\Psi_{\mu,n}$ be an internal set consisting of all internal mappings $\psi : H \to M'$ with the following properties:

(i) H is a hyperfinite subset of M containing x_μ;

(ii) $\psi(x_\mu) = y_\mu$;

(iii) $|d'(\psi(x), \psi(y)) - d(x, y)| < 1/(n + 1)$ for all $(x, y) \in H^2$.

If $F \subseteq D$ is finite, then the mapping $\xi_F : \{x_\mu : \mu \in F\} \to M'$, $\xi_F(x_\mu) = y_\mu$, is in $\Psi_{\mu,n}$ for every $(\mu, n) \in F \times \mathbb{N}$. By κ-saturation there exists a mapping $\psi_0 : H_0 \to M'$ which belongs to the intersection of all $\Psi_{\mu,n}$. Since D is dense in $\mathcal{M}$, by ω_1-saturation H_0 intersects with every $\mu \in \mathcal{M}$. Since H_0 is hyperfinite, by transfer there exists an internal mapping $r : M \to H_0$ such that

$$d(x, r(x)) = \min\{d(x, y) : y \in H_0\}$$

for every $x \in M$; it follows that $x \in \mu \in \mathcal{M}$ implies $r(x) \in \mu$. It is easily checked that $\phi = \psi_0 \circ r$ is a uniform lifting of some mapping $g : \mathcal{M} \to \mathcal{H}'$. Since g is uniformly liftable, it is continuous. Mappings f and g coincide on D, and since D is dense in $\mathcal{M}$, we have $f = g$. Thus ϕ is a uniform lifting of f.

Now we turn to the general case. Since $|\mathcal{M}| < |V(\Xi)|$, there exist isometries $h : \mathcal{M} \to X$ and $h' : f[\mathcal{M}] \to X'$, where metric spaces X and X' are in $V(\Xi)$. Let e and e' denote the canonical embeddings of X and X' into their nonstandard hulls. Let $\psi : M \to {}^*X$ be a uniform lifting of an isometry $e \circ h : \mathcal{M} \to e[X]$. Since f is continuous, $f[\mathcal{M}]$ has a dense subspace of cardinality $< \kappa$. Therefore, there exists a uniform lifting $\tilde{\psi} : {}^*X' \to M'$ of an isometry $(e' \circ h')^{-1} : e'[X'] \to f[\mathcal{M}]$. It is clear that

$$\phi = \tilde{\psi} \circ {}^*(h' \circ f \circ h^{-1}) \circ \psi$$

is a uniform lifting of f. □

The condition $|\mathcal{M}| < |V(\Xi)|$ in Proposition 1.2 is redundant, see Theorem 1.8 below.

See Henson and Moore [11] and Baratella and Ng [2] for some related results in the context of nonstandard hulls of *normed spaces.

Our goal is to extend Proposition 1.2 to a more general class of spaces that includes other types of nonstandard hulls, in particular, nonstandard hulls of topological vector spaces [10]. Following Gordon [7], we consider topological spaces obtained as subspaces of quotients of internal sets by $\Pi^0_1(\kappa)$ equivalence relations. (A set is said to be $\Pi^0_1(\kappa)$ (resp. Π^0_1) if it is the intersection of $<\kappa$ (resp. $\leq\omega$) internal sets.) If R is a $\Pi^0_1(\kappa)$ equivalence relation on internal set M, then for every $\mathcal{M}\subseteq M/R$ and every $Z\subseteq M$ we put

$$O_{\mathcal{M}}(Z)=\{\mu\in\mathcal{M}:\ \mu\subseteq Z\},$$

$$\mathrm{st}_{\mathcal{M}}(Z)=\{\mu\in\mathcal{M}:\ \mu\cap Z\neq\emptyset\}.$$

An easy argument shows that

$$\{O_{M/R}(T):\quad T\subseteq M \text{ is internal}\}$$

is the base of some Hausdorff topology on M/R. (Note that all equivalence classes $\mu\in M/R$ are $\Pi^0_1(\kappa)$ and apply κ-saturation.) We call this topology the *canonical topology* on M/R. It follows from Proposition 2.4 below that this topology is generated by some family of pseudometrics and hence is Tychonoff.

Example 1.3 Let $(\mathcal{H},d^\circ_c)$ be the nonstandard hull of *metric space (M,d) around $c\in M$. Then

$$R=d^{-1}(\mu_{\mathbb{R}}(0))=\bigcap_{n\in\mathbb{N}}\{(x,y)\in M^2:\ d(x,y)<1/(n+1)\}$$

is a Π^0_1 equivalence relation on M and $\mathcal{H}\subseteq M/R$. We claim that the topology generated by d°_c coincides with that induced by the canonical topology on M/R. Fix $\mu\in\mathcal{H}$ and $x\in\mu$; put

$$b(x,n)=\{y\in M:\ d(x,y)<1/(n+1)\},$$

$$B(\mu,n)=\{\nu\in\mathcal{H}:\ d^\circ_c(\mu,\nu)<1/(n+1)\}.$$

If $T\subseteq M$ is internal and $\mu\subseteq T$, then by ω_1-saturation there exists $n\in\mathbb{N}$ so that $\mu\subseteq b(x,n)\subseteq T$, and it follows that $\mu\in B(\mu,n)\subseteq O_{M/R}(T)\cap\mathcal{H}$. On the other hand, $\mu\in O_{M/R}(b(x,n+1))\subseteq B(\mu,n)$ for every $n\in\mathbb{N}$, and the proof of our claim is complete.

Example 1.4 Let (K,τ) be a compact Hausdorff space, $K\in V(\Xi)$. Assume that there exists a base $\mathcal{B}$ of topology τ of cardinality $<\kappa$. It is well-known that ${}^*K$ is the disjoint union of monads $\mu_K(x)$, $x\in K$; let $R=\bigcup_{x\in K}\mu_K(x)^2$ be the corresponding equivalence relation. Let P denote the family of all pairs $(U,V) \in \mathcal{B}^2$ such that $\operatorname{cl} U \subseteq V$, where $\operatorname{cl} U$ is the closure of U in K. We have

$$R = \bigcap_{(U,V)\in P} {}^*V^2 \cup {}^*(K \setminus \operatorname{cl} U)^2,$$

therefore, R is $\Pi^0_1(\kappa)$. It is easily checked that the mapping e defined by $e(x) = \mu_K(x)$ is a homeomorphism of K onto ${}^*K/R$ endowed with the canonical topology.

Example 1.5 Let (X,τ) be a Tychonoff space, $X \in V(\Xi)$. Assume that there exists a base $\mathcal{B}$ of topology τ of cardinality $< \kappa$. By [4, Theorem 3.5.2] there exists a compact Hausdorff space (K,τ') containing (X,τ) as a subspace and such that there exists a base of topology τ' of cardinality $< \kappa$. By [4, Theorem 3.5.3] we may also assume that $K \in V(\Xi)$. By the previous example $R_K = \bigcup_{x\in K} \mu_K(x)^2$ is $\Pi^0_1(\kappa)$, therefore, the equivalence relation $R = R_K \cap {}^*X^2$ on ${}^*X$ is also $\Pi^0_1(\kappa)$. For every $x \in X$ we have $\mu_X(x) = {}^*X \cap \mu_K(x)$, hence $\mu_X(x) \in {}^*X/R$. As in the case of compact spaces, the mapping $e : X \to {}^*X/R$, $e(x) = \mu_X(x)$, is a homeomorphism of X onto $\{\mu_X(x) : x \in X\}$.

Example 1.6 Let $E \in V(\Xi)$ be a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and τ a Hausdorff topology on E such that (E,τ) is a topological vector space. Assume that the filter of neighborhoods of $0 \in E$ has a base $\mathcal{B}_0$ of cardinality $< \kappa$. Then

$$R = \{(x,y) \in {}^*E^2 : x - y \in \mu_E(0)\} = \bigcap_{U\in\mathcal{B}_0} \{(x,y) \in {}^*E^2 : x - y \in {}^*U\}$$

is a $\Pi^0_1(\kappa)$ equivalence relation on ${}^*E$. The nonstandard hull of (E,τ) is a topological vector space $(\hat{E},\hat{\tau})$ defined by

$$\hat{E} = \{\mu \in {}^*E/R : \mu \subseteq \operatorname{fin}({}^*E)\},$$

where

$$\operatorname{fin}({}^*E) = \{x \in {}^*E : \epsilon x \in \mu_E(0) \text{ for every } \epsilon \in \mu_{\mathbb{F}}(0)\};$$

the filter of neighborhoods of $\mu_E(0)$ in $\hat{E}$ is generated by the sets $\operatorname{st}_{\hat{E}}({}^*U)$, $U \in \mathcal{B}_0$, see Henson [10]. It is easily checked that the nonstandard hull topology $\hat{\tau}$ coincides with that induced by the canonical topology on ${}^*E/R$. (Apply Lemma 2.5 below.)

For further examples see Luxemburg [12], Henson [9], Gordon [7, 8], Mlček and Zlatoš [13], and Ziman and Zlatoš [14].

Let R (resp. R') be a $\Pi^0_1(\kappa)$ equivalence relation on an internal set M (resp. M') and f a mapping from $\mathcal{M} \subseteq M/R$ into M'/R'. We say that a mapping $\phi : Z \to M'$ is a *weak uniform lifting* of f if ϕ is an internal mapping with $Z = \operatorname{dom}\phi \subseteq M$ such that $\mu \cap Z \neq \emptyset$ and $\phi[\mu \cap Z] \subseteq f(\mu)$ for every $\mu \in \mathcal{M}$. A mapping ϕ is said to be a *uniform lifting* of f if ϕ is a weak uniform lifting of f with $\operatorname{dom}\phi = M$. We say that f is *(weakly) uniformly liftable* if there exists a (weak) uniform lifting of f. Every weakly uniformly liftable mapping is continuous, see Proposition 4.2 below.

Gordon [7, Theorem 1.2] proved that if R is a $\Pi^0_1(\kappa)$ equivalence relation on an internal set M such that M/R is compact, then every continuous mapping $f : M/R \to \operatorname{fin}({}^*\mathbb{C})/\approx$ is uniformly liftable.

We say that a topological space is *κ-separable* if it has a dense subspace of cardinality $< \kappa$.

The following are our main results:

Theorem 1.7 *Let R (resp. R') be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M (resp. M') and $\mathcal{M}$ a κ-separable subspace of M/R. Then every continuous mapping $f : \mathcal{M} \to M'/R'$ is weakly uniformly liftable. Moreover, a weak uniform lifting $\phi : H \to M'$ of f can be chosen so that $H = \operatorname{dom}\phi$ is a hyperfinite subset of M.*

Theorem 1.8 *Let R (resp. R') be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M (resp. M') and $\mathcal{M}$ a subspace of M/R. Assume that at least one of the following conditions is satisfied:*

(1) *$\mathcal{M}$ is κ-separable and R is Π^0_1;*

(2) *$\mathcal{M}$ is Lindelöf and R' is Π^0_1;*

(3) *$\mathcal{M}$ is compact and $\mathcal{M} = M/R$;*

(4) *$\mathcal{M}$ is separable and metrizable.*

Then every continuous mapping $f : \mathcal{M} \to M'/R'$ is uniformly liftable.

Also we show that each of the following conditions (in the context of Theorem 1.8) is not sufficient for the existence of uniform liftings:

(i) $\mathcal{M}$ is separable and $f : \mathcal{M} \to {}^*[0, 1]/\approx$ is continuous;

(ii) $\mathcal{M}$ is separable and compact and $f : \mathcal{M} \to \mathcal{M}'$ is a homeomorphism;

see Example 4.6 below.

2 Generating families of *pseudometrics

Let X be an arbitrary set and let U and V be the subsets of X^2. Recall the following definitions:

$$U^{-1} = \{(x,y) \in X^2 : (y,x) \in U\},$$

$$U[x] = \{y \in X : (x,y) \in U\},$$

$$V \circ U = \{(x,z) \in X^2 : \text{ there exists } y \in X \text{ so that } (x,y) \in U \text{ and } (y,z) \in V\},$$

$$U^{(1)} = U, \quad U^{(n+1)} = U^{(n)} \circ U, \quad n \geq 1.$$

By Δ_X we denote the diagonal of X.

Lemma 2.1 *Let X be a set and $V_0 = X^2, V_1, V_2, \ldots$ a sequence of sets such that $\Delta_X \subseteq V_n = V_n^{-1} \subseteq X^2$ and $V_{n+1}^{(3)} \subseteq V_n$ for all $n \geq 1$. Then there exists a pseudometric d on X such that $d^{-1}([0, 1/2^n)) \subseteq V_n \subseteq d^{-1}([0, 1/2^n])$ for all $n \geq 1$.*

Proof This is an easy corollary of [4, Theorem 8.1.10]. □

Let R be an equivalence relation on internal set M and $\{d_\alpha : \alpha \in A\}$ a family of *pseudometrics on M. We say that R is *generated by* $\{d_\alpha : \alpha \in A\}$ and that $\{d_\alpha : \alpha \in A\}$ is a *generating family* for R if

$$R = \{(x,y) \in M^2 : d_\alpha(x,y) \approx 0 \text{ for every } \alpha \in A\}.$$

Proposition 2.2 *Every $\Pi_1^0(\kappa)$ equivalence relation R on internal set M has a generating family $\{d_\alpha : \alpha \in A\}$ of *pseudometrics on M such that $|A| < \kappa$.*

Proof Let $\mathcal{U}$ be an internal set consisting of all internal sets U such that $\Delta_M \subseteq U = U^{-1} \subseteq M^2$. Let $\{U_\alpha : \alpha \in A\} \subseteq \mathcal{U}$ be such that $|A| < \kappa$ and $R = \bigcap_{\alpha \in A} U_\alpha$. We may assume that $\{U_\alpha : \alpha \in A\}$ is closed under finite intersections. Clearly, $R \subseteq \bigcap_{\alpha \in A} U_\alpha^{(2)}$. On the other hand, if $(x,y) \in U_\alpha^{(2)}$ for all $\alpha \in A$, then by κ-saturation

$$R[x] \cap R[y] = \bigcap_{(\alpha,\beta) \in A^2} U_\alpha[x] \cap U_\beta[y] \neq \emptyset,$$

hence $(x,y) \in R$. It follows that $R = \bigcap_{\alpha \in A} U_\alpha^{(2)}$. By κ-saturation for every $\alpha \in A$ there exists $\beta \in A$ such that $U_\beta^{(2)} \subseteq U_\alpha$. Hence for every $\alpha \in A$ there exists $\gamma \in A$ such that $U_\gamma^{(3)} \subseteq U_\gamma^{(4)} \subseteq U_\alpha$. Let $f : A \to A$ be such that $U_{f(\alpha)}^{(3)} \subseteq U_\alpha$ for every $\alpha \in A$.

Fix $\alpha \in A$. By ω_1-saturation there exists an internal sequence $(V_\nu)_{0\leq\nu\leq H}$ consisting of elements of $\mathcal{U}$ such that $H \in {}^*\mathbb{N}\setminus\mathbb{N}$, $V_0 = M^2$, $V_1 = U_\alpha$, $V_{n+1} = U_{f^{(n)}(\alpha)}$ for every $n \in \mathbb{N}\setminus\{0\}$, and $V_{\nu+1}^{(3)} \subseteq V_\nu$ for every $\nu < H$. Put $V_\nu = \Delta_M$ for all $\nu > H$. By transfer of Lemma 2.1 there exists a *pseudometric d_α on M such that

$$d_\alpha^{-1}([0,1/2^\nu)_{{}^*\mathbb{R}}) \subseteq V_\nu \subseteq d_\alpha^{-1}([0,1/2^\nu]_{{}^*\mathbb{R}}), \quad \nu \geq 1.$$

We have $R \subseteq d_\alpha^{-1}(\mu_\mathbb{R}(0)) \subseteq U_\alpha$. It is clear that R is generated by the family $\{d_\alpha : \alpha \in A\}$. □

Proposition 2.3 *Every Π^0_1 equivalence relation on internal set M is generated by some *metric d on M.*

Proof Applying Proposition 2.2 with $\kappa = \omega_1$, we obtain a generating family $\{d_n : n \in \mathbb{N}\}$ for R. We may assume that all *pseudometrics d_n take values in ${}^*[0,1]$ (e.g., we may consider a *pseudometric $\bar{d}_n(x,y) = \min\{d_n(x,y),1\}$ instead of d_n). Let $(d_\nu)_{0\leq\nu\leq H}$ be an internal extension of $(d_n)_{n\in\mathbb{N}}$ which also consists of *pseudometrics on M with values in ${}^*[0,1]$. We may assume that d_H is a $\{0,1\}$-valued *metric on M. Put

$$d(x,y) = \sum_{\nu=0}^{H} d_\nu(x,y)/2^{\nu+1}.$$

Clearly, $d^{-1}(\mu_\mathbb{R}(0)) \subseteq R$. On the other hand, if $(x,y) \in R$, then by ω_1-saturation there exists $K \in {}^*\mathbb{N}\setminus\mathbb{N}$, $K < H$, so that $\sum_{\nu=0}^{K} d_\nu(x,y)/2^{\nu+1} \approx 0$; since then

$$\sum_{\nu=K+1}^{H} d_\nu(x,y)/2^{\nu+1} \leq 1/2^{K+1} \approx 0,$$

we obtain $d(x,y) \approx 0$. □

Assume that an equivalence relation R on internal set M is generated by the family $\{d_\alpha : \alpha \in A\}$ of *pseudometrics. For every $\alpha \in A$ define $d_\alpha^\circ : (M/R)^2 \to [0,\infty]$ by

$$d_\alpha^\circ(\mu,\nu) = {}^\circ(d_\alpha(x,y)), \quad x \in \mu,\ y \in \nu;$$

it is easily checked that d_α° is a $[0,\infty]$-valued pseudometric on M/R.

Proposition 2.4 *Let R be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M and $\{d_\alpha : \alpha \in A\}$ a generating family of *pseudometrics for R with $|A| < \kappa$. Then the topology on M/R generated by pseudometrics d_α°, $\alpha \in A$, coincides with the canonical topology.*

Proof Fix $x \in \mu \in M/R$ and put

$$b(x, n, \alpha) = \{y \in M : d_\alpha(x, y) < 1/(n+1)\},$$

$$B(\mu, n, \alpha) = \{\nu \in M/R : d^\circ_\alpha(\mu, \nu) < 1/(n+1)\}.$$

If $\mu \subseteq T$, where T is an internal subset of M, then since

$$\mu = \bigcap_{(n,\alpha)\in\mathbb{N}\times A} b(x, n, \alpha) \subseteq T,$$

by κ-saturation there exists a finite set $F \subseteq \mathbb{N} \times A$ with $\bigcap_{(n,\alpha)\in F} b(x, n, \alpha) \subseteq T$. It follows that

$$\mu \in \bigcap_{(n,\alpha)\in F} B(\mu, n, \alpha) \subseteq O_{M/R}(T).$$

On the other hand, for any finite $F \subseteq \mathbb{N} \times A$ we have

$$\mu \in O_{M/R}\Big(\bigcap_{(n,\alpha)\in F} b(x, n+1, \alpha)\Big) \subseteq \bigcap_{(n,\alpha)\in F} B(\mu, n, \alpha).$$

The result follows. □

Lemma 2.5 *Let R be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M and U an open in M/R set containing $\mu \in M/R$. Then there exists an internal set T such that $\mu \subseteq T \subseteq M$ and $\mathrm{st}_{M/R}(T) \subseteq U$.*

Proof Easy (see the proof of Proposition 2.4). □

Lemma 2.6 *Let R (resp. R') be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M (resp. M'). Then*

$$R \otimes R' = \{((x, x'), (y, y')) \in (M \times M')^2 : (x, y) \in R \text{ and } (x', y') \in R'\}$$

is a $\Pi^0_1(\kappa)$ equivalence relation on $M \times M'$ and the mapping $h : (\mu, \mu') \mapsto \mu \times \mu'$ is a homeomorphism of $(M/R) \times (M'/R')$ onto $(M \times M')/(R \otimes R')$.

Proof Let $\{d_\alpha : \alpha \in A\}$ and $\{d'_\beta : \beta \in B\}$ be generating families of *pseudometrics for R and R' respectively with $|A| < \kappa$ and $|B| < \kappa$. Put

$$\rho_\alpha((x, x'), (y, y')) = d_\alpha(x, y), \quad \alpha \in A,$$

$$\rho'_\beta((x, x'), (y, y')) = d'_\beta(x', y'), \quad \beta \in B.$$

Since $R \otimes R'$ is generated by $\{\rho_\alpha : \alpha \in A\} \cup \{\rho'_\beta : \beta \in B\}$, it is $\Pi^0_1(\kappa)$. Using Proposition 2.4 and noting that

$$\rho^\circ_\alpha(\mu \times \mu', \nu \times \nu') = d^\circ_\alpha(\mu, \nu), \quad \alpha \in A,$$
$$\rho'^\circ_\beta(\mu \times \mu', \nu \times \nu') = d'^\circ_\beta(\mu', \nu'), \quad \beta \in B,$$

it is easy to check that h is a homeomorphism. □

The following important lemma will be used without mentioning:

Lemma 2.7 *Let $\mathcal{M} \subseteq (M/R) \cap (M/\tilde{R})$, where R and $\tilde{R}$ are $\Pi^0_1(\kappa)$ equivalence relations on internal set M. Then the canonical topologies on M/R and $M/\tilde{R}$ induce the same topology on $\mathcal{M}$.*

Proof Note that $\mathcal{M} \cap O_{M/R}(T) = O_{\mathcal{M}}(T) = \mathcal{M} \cap O_{M/\tilde{R}}(T)$ for every internal $T \subseteq M$. □

3 κ-Separable subspaces and weak uniform liftings

Lemma 3.1 *Let R be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M and $\mathcal{M}$ a subspace of M/R. If $\mathcal{M}$ is Lindelöf, then it is κ-separable. If $\mathcal{M}$ is compact, then $\bigcup \mathcal{M} = \bigcup_{\mu \in \mathcal{M}} \mu$ is a $\Pi^0_1(\kappa)$ set and there exists a hyperfinite set $H \subseteq M$ such that $\mathcal{M} = \mathrm{st}_{M/R}(H)$.*

Proof Assume that $\mathcal{M}$ is Lindelöf (resp. compact). Let $\{U_\alpha : \alpha \in A\}$ be a family of internal subsets of M^2 such that $|A| < \kappa$ and $R = \bigcap_{\alpha \in A} U_\alpha$. We may assume that $U_\alpha^{-1} = U_\alpha$ for every $\alpha \in A$ and that $\{U_\alpha : \alpha \in A\}$ is closed under finite intersections. Choose $x_\mu \in \mu$ for every $\mu \in \mathcal{M}$. For every $\alpha \in A$ the sets $O_{\mathcal{M}}(U_\alpha[x_\mu])$, $\mu \in \mathcal{M}$, are open in $\mathcal{M}$ and cover $\mathcal{M}$, therefore, there exists a countable (resp. finite) set $\mathcal{C}_\alpha \subseteq \mathcal{M}$ with $\mathcal{M} = \bigcup_{\mu \in \mathcal{C}_\alpha} O_{\mathcal{M}}(U_\alpha[x_\mu])$.

Put $D = \bigcup_{\alpha \in A} \mathcal{C}_\alpha$; note that D has cardinality $< \kappa$. We claim that D is dense in $\mathcal{M}$. Fix $\mu \in \mathcal{M}$ and an open in $\mathcal{M}$ neighborhood U of μ. By Lemma 2.5 there exists an internal set $T \subseteq M$ such that $\mu \subseteq T$ and $\mathrm{st}_{\mathcal{M}}(T) \subseteq U$. By κ-saturation there exists $\alpha \in A$ such that $U_\alpha[x_\mu] \subseteq T$. There exists $\nu \in \mathcal{C}_\alpha$ with $\mu \in O_{\mathcal{M}}(U_\alpha[x_\nu])$. It follows that $x_\nu \in U_\alpha[x_\mu]$ and hence $\nu \in U$. Thus $\mathcal{M}$ is κ-separable.

Now we assume that $\mathcal{M}$ is compact. For every $\alpha \in A$ put $W_\alpha = \bigcup_{\mu \in \mathcal{C}_\alpha} U_\alpha[x_\mu]$. Clearly, $\bigcup \mathcal{M} \subseteq W_\alpha$. Since $\mathcal{C}_\alpha$ is finite, W_α is internal. We claim that $\bigcup \mathcal{M} =$

$\bigcap_{\alpha\in A} W_\alpha$. Let $y \in \nu \in (M/R) \setminus \mathcal{M}$. Since $\mathcal{M}$ is closed in M/R, by Lemma 2.5 and κ-saturation there exists $\alpha \in A$ so that $U_\alpha[y] \cap \bigcup \mathcal{M} = \emptyset$, hence $y \notin W_\alpha$. Thus $\bigcup \mathcal{M}$ is $\Pi^0_1(\kappa)$.

Since $\{x_\mu : \mu \in D\} \subseteq \bigcup \mathcal{M}$, $|D| < \kappa$, and $\bigcup \mathcal{M}$ is $\Pi^0_1(\kappa)$, by κ-saturation there exists a hyperfinite set H with $\{x_\mu : \mu \in D\} \subseteq H \subseteq \bigcup \mathcal{M}$. Then $\mathrm{st}_{M/R}(H)$ is a closed subset of M/R such that $D \subseteq \mathrm{st}_{M/R}(H) \subseteq \mathcal{M}$; since D is dense in $\mathcal{M}$, $\mathrm{st}_{M/R}(H) = \mathcal{M}$. □

If T and S are internal sets, then by S^T_{int} we denote the internal set consisting of all internal mappings $\phi : T \to S$.

Lemma 3.2 *Let T be an internal set and A a nonempty subset of T of cardinality $< \kappa$. Then the space ${}^*[0,1]^T_{\mathrm{int}}/R$, where R is generated by *pseudometrics $d_\alpha(x,y) = |x(\alpha) - y(\alpha)|$, $\alpha \in A$, is compact.*

Proof We claim that ${}^*[0,1]^T_{\mathrm{int}}/R$ is homeomorphic to $[0,1]^A$. Clearly, the mapping

$$w : {}^*[0,1]^T_{\mathrm{int}} \to [0,1]^A, \quad (w(x))(\alpha) = {}^\circ(x(\alpha)),$$

induces an injection $h : {}^*[0,1]^T_{\mathrm{int}}/R \to [0,1]^A$. By κ-saturation h is onto. The usual topology on $[0,1]^A$ is generated by pseudometrics $\rho_\alpha(f,g) = |f(\alpha) - g(\alpha)|$, $\alpha \in A$, and since $d^\circ_\alpha(\mu,\nu) = \rho_\alpha(h(\mu),h(\nu))$, by Proposition 2.4 mapping h is a homeomorphism. □

Theorem 3.3 *Let $\mathcal{M}$ be a κ-separable subspace of M/R, where R is a $\Pi^0_1(\kappa)$ equivalence relation on internal set M. Then there exists a $\Pi^0_1(\kappa)$ equivalence relation $\tilde{R}$ on M such that $M/\tilde{R}$ is compact and $\mathcal{M} \subseteq M/\tilde{R}$.*

Proof Let $\{d_\alpha : \alpha \in A\}$ be a generating family of *pseudometrics for R such that $|A| < \kappa$ (see Proposition 2.2). We may assume that all d_α take values in ${}^*[0,1]$. Let D be a dense subspace of $\mathcal{M}$ of cardinality $< \kappa$. Choose $x_\mu \in \mu$ for every $\mu \in D$ and put $f_{\alpha,\mu}(x) = d_\alpha(x, x_\mu)$. Let $\tilde{R}$ be a $\Pi^0_1(\kappa)$ equivalence relation on M generated by *pseudometrics

$$\sigma_{\alpha,\mu}(x,y) = |f_{\alpha,\mu}(x) - f_{\alpha,\mu}(y)|, \quad (\alpha,\mu) \in A \times D.$$

Put $\mathcal{F} = {}^*[0,1]^M_{\mathrm{int}}$. Let R' be a $\Pi^0_1(\kappa)$ equivalence relation on ${}^*[0,1]^{\mathcal{F}}_{\mathrm{int}}$ generated by *pseudometrics

$$\rho_{\alpha,\mu}(\phi,\chi) = |\phi(f_{\alpha,\mu}) - \chi(f_{\alpha,\mu})|, \quad (\alpha,\mu) \in A \times D.$$

By Lemma 3.2 $\mathcal{K} = {}^*[0,1]^{\mathcal{F}}_{\mathrm{int}}/R'$ is compact.

Let $\theta : M \to {}^*[0,1]^{\mathcal{F}}_{\mathrm{int}}$ be an internal mapping defined by $(\theta(x))(f) = f(x)$. Then $\mathcal{K}_0 = \mathrm{st}_{\mathcal{K}}(\theta[M])$ is a closed subspace of $\mathcal{K}$ and therefore is compact. Note that $\sigma_{\alpha,\mu}(x,y) = \rho_{\alpha,\mu}(\theta(x),\theta(y))$ for every $(\alpha,\mu) \in A \times D$. Using Proposition 2.4, it is easy to check that θ induces a homeomorphism of $M/\tilde{R}$ onto $\mathcal{K}_0$. It follows that $M/\tilde{R}$ is compact.

In order to prove that $\mathcal{M} \subseteq M/\tilde{R}$, note that $(x,y) \in R$ implies $d_\alpha(x,x_\mu) \approx d_\alpha(y,x_\mu)$ and hence $\sigma_{\alpha,\mu}(x,y) \approx 0$ for every $(\alpha,\mu) \in A \times D$. On the other hand, if $x \in \mu \in \mathcal{M}$ and $y \in M \setminus \mu$, then $d_\alpha(x,y) > \epsilon$ for some $\alpha \in A$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$; since D is dense in $\mathcal{M}$, there exists $\mu \in D$ such that $d_\alpha(x,x_\mu) < \epsilon/3$; then $d_\alpha(x_\mu,y) > \epsilon/2$; therefore, as $\sigma_{\alpha,\mu}(x,y) \not\approx 0$, we have $(x,y) \notin \tilde{R}$. □

Corollary 3.4 *Let $\mathcal{M}$ be a separable subspace of some nonstandard hull $\mathcal{H}$ of a *metric space (M,d). Then there exists a *metric $\tilde{d} : M^2 \to {}^*[0,1]$ on M such that the nonstandard hull $\tilde{\mathcal{H}}$ of $(M,\tilde{d})$ is compact and $\mathcal{M} \subseteq \tilde{\mathcal{H}}$.*

Proof Put $\kappa = \omega_1$ in Theorem 3.3 and apply Propositions 2.3 and 2.4. □

Corollary 3.5 *Let R be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M and $\mathcal{M}$ a κ-separable subspace of M/R. Then for every closed in $\mathcal{M}$ set $\mathcal{A} \subseteq \mathcal{M}$ there exists a hyperfinite set $H \subseteq M$ such that $\mathrm{st}_{\mathcal{M}}(H) = \mathcal{A}$.*

Proof This follows easily from Theorem 3.3 and Lemma 3.1. □

Proof of Theorem 1.7 Since $\mathcal{M}$ is κ-separable and f is continuous, $f[\mathcal{M}]$ is also κ-separable. By Theorem 3.3 there exist $\Pi^0_1(\kappa)$ equivalence relations $\tilde{R}$ and $\tilde{R}'$ on M and M' respectively such that $\mathcal{M} \subseteq M/\tilde{R}$, $f[\mathcal{M}] \subseteq M'/\tilde{R}'$, and such that the spaces $M/\tilde{R}$ and $M'/\tilde{R}'$ are compact. Let F be the closure of f in $(M/\tilde{R}) \times (M'/\tilde{R}')$; since f is continuous, we have $F \cap (\mathcal{M} \times (M'/\tilde{R}')) = f$. Since F is compact, by Lemmas 2.6 and 3.1 there exists a hyperfinite set $\tilde{H} \subseteq M \times M'$ such that

$$\tilde{H} \subseteq \bigcup_{(\mu,\mu')\in F} \mu \times \mu'$$

and $\tilde{H} \cap (\mu \times \mu') \neq \emptyset$ for every $(\mu,\mu') \in F$. Let $H = \mathrm{dom}\,\tilde{H}$. By transfer there exists an internal mapping $\phi : H \to M'$ such that $(x,\phi(x)) \in \tilde{H}$ for every $x \in H$; it is clear that ϕ is a weak uniform lifting of f. □

4 Uniform liftings

Proposition 4.1 *Let R (resp. R') be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M (resp. M'), $\mathcal{M}$ a subspace of M/R, and $f : \mathcal{M} \to M'/R'$ a mapping. Consider the following statements:*

(1) *f is uniformly liftable;*

(2) *there exists an extension $\tilde{f} : M/R \to M'/R'$ of f which is continuous at every $\mu \in \mathcal{M}$.*

We have (1) $\Rightarrow$ (2) *and if M/R and M'/R' are compact, then* (1) *and* (2) *are equivalent.*

Proof Assume (1). Let $\phi : M \to M'$ be a uniform lifting of f. Put

$$\Phi = \{(\mu, \mu') \in (M/R) \times (M'/R') : \ \phi \cap (\mu \times \mu') \neq \emptyset\};$$

note that $\operatorname{dom} \Phi = M/R$ and $\Phi \cap (\mathcal{M} \times (M'/R')) = f$. Let $\tilde{f} : M/R \to M'/R'$ be an arbitrary mapping such that $\tilde{f} \subseteq \Phi$. Clearly, $\tilde{f}$ is an extension of f. We claim that $\tilde{f}$ is continuous at every $\mu \in \mathcal{M}$. Let U' be an open neighborhood of $f(\mu)$. By Lemma 2.5 there exists an internal set $T' \subseteq M'$ such that $f(\mu) \subseteq T'$ and $\mathrm{st}_{M'/R'}(T') \subseteq U'$. Put $T = \phi^{-1}[T']$. Then $U = O_{M/R}(T)$ is an open neighborhood of μ. Fix $\nu \in U$. There exists $x \in \nu$ such that $\phi(x) \in \tilde{f}(\nu)$. We have $x \in T$, therefore, $\phi(x) \in T'$ and $\tilde{f}(\nu) \in U'$.

Assume (2) and that M/R and M'/R' are compact. Let F be the closure of $\tilde{f}$ in $(M/R) \times (M'/R')$. Since $\tilde{f}$ is continuous at every $\mu \in \mathcal{M}$, we have $F \cap (\mathcal{M} \times (M'/R')) = f$. By Lemmas 2.6 and 3.1 the set

$$\Pi = \bigcup_{(\mu, \mu') \in F} \mu \times \mu'$$

is $\Pi^0_1(\kappa)$. Since $\operatorname{dom} \Pi = M$, by κ-saturation there exists an internal mapping $\phi : M \to M'$ such that $\phi \subseteq \Pi$. It is easily checked that ϕ is a uniform lifting of f. $\square$

Proposition 4.2 *Let R (resp. R') be a $\Pi^0_1(\kappa)$ equivalence relation on internal set M (resp. M') and $\mathcal{M}$ a subspace of M/R. Then every weakly uniformly liftable mapping $f : \mathcal{M} \to M'/R'$ is continuous.*

Proof Let $\phi : Z \to M'$ be a weak uniform lifting of f. Note that $\mathcal{M} \subseteq \mathrm{st}_{M/R}(Z)$. By Propositions 2.2 and 2.4 the canonical bijection

$$h : Z/(R \cap Z^2) \to \mathrm{st}_{M/R}(Z)$$

is a homeomorphism. Since ϕ is a uniform lifting of $f \circ h : h^{-1}[\mathcal{M}] \to M'/R'$, by Proposition 4.1 $f \circ h$ is continuous. Hence f is also continuous. $\square$

Lemma 4.3 *Let X be a dense subspace of a Hausdorff space $\tilde{X}$, Y a subspace of a compact Hausdorff space $\tilde{Y}$, and $f : X \to Y$ a continuous mapping. Then there exists an extension $\tilde{f} : \tilde{X} \to \tilde{Y}$ of f which is continuous at every point of X.*

Proof (It is convenient to use nonstandard analysis at every step.) Let F be the closure of f in $\tilde{X} \times \tilde{Y}$. Since X is dense in $\tilde{X}$ and $\tilde{Y}$ is compact, $\operatorname{dom} F = \tilde{X}$. Since f is continuous, $F \cap (X \times \tilde{Y}) = f$. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be any mapping such that $\tilde{f} \subseteq F$; then $\tilde{f}$ is continuous at every point of X. □

Lemma 4.4 *Let X be a nonempty closed subspace of metrizable space $\tilde{X}$. Then there exists a mapping $r : \tilde{X} \to X$ such that $r(x) = x$ for every $x \in X$ and which is continuous at every point of X.*

Proof Let d be any compatible metric on $\tilde{X}$. For every $y \in \tilde{X}$ choose $r(y) \in X$ so that $d(y, r(y)) \leq 2d(y, X)$. □

Lemma 4.5 *Let X be a Lindelöf subspace of a Tychonoff space $\tilde{X}$ and Y a subspace of a compact Hausdorff space $\tilde{Y}$. Assume that at least one of the spaces X and Y is separable and metizable. Then for every continuous mapping $f : X \to Y$ there exists an extension $\tilde{f} : \tilde{X} \to \tilde{Y}$ which is continuous at every point of X.*

Proof In both cases there exist continuous mappings $g : X \to Z$ and $h : Z \to Y$ such that $Z \subseteq [0,1]^{\mathbb{N}}$ and $h \circ g = f$. (Note that every separable metrizable space embeds into $[0,1]^{\mathbb{N}}$.) Since X is Lindelöf, by [1, Corollary 16] for every $n \in \mathbb{N}$ there exists an extension $\tilde{g}_n : \tilde{X} \to \mathbb{R}$ of $g_n = \mathrm{pr}_n \circ g$ which is continuous at every point of X. We may assume that $\tilde{g}_n : \tilde{X} \to [0,1]$. Let $\tilde{g} : \tilde{X} \to [0,1]^{\mathbb{N}}$ be such that $\tilde{g}_n = \mathrm{pr}_n \circ \tilde{g}$ for every $n \in \mathbb{N}$; it is easily checked that $\tilde{g}$ is an extension of g which is continuous at every point of X.

Let K be the closure of Z in $[0,1]^{\mathbb{N}}$. By Lemma 4.4 there exists a mapping $r : [0,1]^{\mathbb{N}} \to K$ such that $r(t) = t$ for every $t \in K$ and which is continuous at every point of K. By Lemma 4.3 we obtain an extension $\tilde{h} : K \to \tilde{Y}$ of h which is continuous at every point of Z. It is clear that $\tilde{f} = \tilde{h} \circ r \circ \tilde{g}$ is an extension of f which is continuous at every point of X. □

Proof of Theorem 1.8 Case (1): By Theorem 1.7 there exists a weak uniform lifting $\psi : H \to M'$ of f such that H is a hyperfinite subset of M. By Proposition 2.3 there exists a *metric d on M with $R = d^{-1}(\mu_{\mathbb{R}}(0))$. Since H is hyperfinite, by transfer there exists an internal mapping $r : M \to H$ such that $d(x, r(x)) = \min\{d(x, y) : y \in H\}$ for every $x \in M$. It is clear that $\phi = \psi \circ r$ is a uniform lifting of f.

Cases (2)-(4): Since $\mathcal{M}$ is κ-separable (apply Lemma 3.1 in cases (2) and (3)), $f[\mathcal{M}]$ is also κ-separable. By Theorem 3.3 there exist $\Pi^0_1(\kappa)$ equivalence relations $\tilde{R}$ and $\tilde{R}'$ on M and M' respectively such that $\mathcal{M} \subseteq M/\tilde{R}$, $f[\mathcal{M}] \subseteq M'/\tilde{R}'$, and such that the spaces $M/\tilde{R}$ and $M'/\tilde{R}'$ are compact. In view of Proposition 4.1, it suffices to check that there exists an extension $\tilde{f} : M/\tilde{R} \to M'/\tilde{R}'$ of f which is continuous at every $\mu \in \mathcal{M}$. In case (3) there is nothing to prove. In cases (2) and (4) we apply Lemma 4.5. In both cases $\mathcal{M}$ is Lindelöf (note that every separable metrizable space is Lindelöf). In case (2) $f[\mathcal{M}]$ is Lindelöf (as a continuous image of Lindelöf space) and M'/R' is metrizable (by Propositions 2.3 and 2.4), therefore, $f[\mathcal{M}]$ is separable and metrizable (as a Lindelöf subspace of metrizable space). □

Example 4.6 Let X be a separable Tychonoff space which is not Lindelöf [4, Examples 3.8.13]. Assume also that X is not almost compact (e.g., consider $X \times \{0, 1\}$ instead of X). (A Tychonoff space X is said to be almost compact if $|\beta X \setminus X| \leq 1$.) By [3, Theorem 1.3] there exist a Tychonoff space Y containing X as a subspace and a continuous function $f : X \to \mathbb{R}$ such that there is no extension $\tilde{f} : Y \to \mathbb{R}$ of f which is continuous at every point of X. By [3, remark on p. 908] we may assume that $f : X \to [0, 1]$. Let K be the closure of X in βY; note that K is separable and compact. Assume that our nonstandard universe $(V(\Xi), V(^*\Xi), *)$ is κ-saturated with $\kappa > 2^{|\beta Y|}$ and also that $\beta Y \in V(\Xi)$. We claim that the following continuous mappings are not uniformly liftable (see Example 1.4):

$$g : \{\mu_{\beta Y}(x) : x \in X\} \to \{\mu_{[0,1]}(r) : r \in [0, 1]\}, \quad \mu_{\beta Y}(x) \mapsto \mu_{[0,1]}(f(x)),$$

$$h : \{\mu_{\beta Y}(y) : y \in K\} \to \{\mu_K(y) : y \in K\}, \quad \mu_{\beta Y}(y) \mapsto \mu_K(y).$$

Assume that g is uniformly liftable; using Proposition 4.1, we obtain that there exists an extension $\hat{f} : \beta Y \to [0, 1]$ of f which is continuous at every point of X; then $\tilde{f} = \hat{f}|Y$ is also continuous at every point of X, a contradiction.

If h is uniformly liftable, then there exists a mapping $r : \beta Y \to K$ such that $r(y) = y$ for every $y \in K$ and which is continuous at every point of K. By Lemma 4.3 there exists an extension $f' : K \to [0, 1]$ of f which is continuous at every point of X. Define $\tilde{f} : Y \to [0, 1]$ by $\tilde{f} = (f' \circ r)|Y$; it is clear that $\tilde{f}$ is an extension of f which is continuous at every point of X, a contradiction.

References

[1] **A V Arhangel'skii**, *Relative topological properties and relative topological spaces*, Topology Appl. 70 (1996) 87–99; doi:10.1016/0166-8641(95)00086-0.

[2] **S Baratella**, **S-A Ng**, *Fixed points in the nonstandard hull of a Banach space*, Nonlinear Analysis 34 (1998) 299–306; doi:10.1016/S0362-546X(98)00571-9.

[3] **A Bella**, **I V Yaschenko**, *Lindelöf property and absolute embeddings*, Proc. Amer. Math. Soc. 127 (1999) 907–913; doi:10.1090/S0002-9939-99-04568-2.

[4] **R Engelking**, *General Topology*, Heldermann Verlag, Berlin (1989).

[5] **S Fajardo**, **H J Keisler**, *Existence theorems in probability theory*, Advances in Math. 120 (1996) 191–257; doi:10.1006/aima.1996.0037.

[6] **S Fajardo**, **H J Keisler**, *Neometric spaces*, Advances in Math. 118 (1996) 134–175; doi:10.1006/aima.1996.0020.

[7] **E I Gordon**, *Nonstandard analysis and compact abelian groups*, Sib. Math. J. 32 (1991) 197–209; doi:10.1007/BF00972766.

[8] **E I Gordon**, *Nonstandard methods in commutative harmonic analysis*, volume 164 of *Translations of mathematical monographs*, AMS (1997).

[9] **C W Henson**, *The nonstandard hulls of a uniform space*, Pacif. J. Math. 43 (1972) 115–137.

[10] **C W Henson**, *Infinitesimals in functional analysis*, from: "Nonstandard analysis and its applications", (N J Cutland, editor), London Mathematical Society Student Texts 10, Cambridge University Press (1988) 140–181.

[11] **C W Henson**, **L C Moore**, *Nonstandard analysis and the theory of Banach spaces*, from: "Nonstandard Analysis–Recent Developments", (A E Hurd, editor), Lecture Notes in Mathematics 983 (1983) 27–112; doi:10.1007/BFb0065334.

[12] **W A J Luxemburg**, *A general theory of monads*, from: "Applications of Model Theory to Algebra, Analysis and Probability", (W A J Luxemburg, editor), Holt, Rinehart and Winston (New York) (1969) 18–86.

[13] **J Mlček**, **P Zlatoš**, *Some Ramsey-type theorems for countably determined sets*, Arch. Math. Logic 41 (2002) 619–630; doi:10.1007/s001530100129.

[14] **M Ziman**, **P Zlatoš**, *Hyperfinite dimensional representations of spaces and algebras of measures*, Monatsh. Math. 148 (2006) 241–261; doi:10.1007/s00605-005-0375-3.

IICO-UASLP, Av. Karakorum 1470, Lomas 4ta Session, San Luis Potosí SLP, 78210 Mexico

appyshchev@hotmail.com

Received: 23 September 2009 Revised: 12 December 2010